§7.2 Riemann Integrable Functions

Thm 7.2.1 (Cauchy Criterion)

$f \in \mathcal{R}[a,b] \Leftrightarrow \forall \varepsilon > 0, \exists \eta_\varepsilon > 0$ such that

if $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ are tagged partitions with

$\|\dot{\mathcal{P}}\| < \eta_\varepsilon$ & $\|\dot{\mathcal{Q}}\| < \eta_\varepsilon$,

then $|S(f, \dot{\mathcal{P}}) - S(f, \dot{\mathcal{Q}})| < \varepsilon$

(Compare : (x_n) converges $\Leftrightarrow \forall \varepsilon > 0, \exists N_\varepsilon > 0$ s.t. if $m, n \geq N_\varepsilon$, $|x_m - x_n| < \varepsilon$)

Pf : ($\Rightarrow$) If $f \in \mathcal{R}[a,b]$ and $L = \int_a^b f$.

Then $\forall \varepsilon > 0$, $\exists \eta_\varepsilon (= \delta_{\varepsilon/2}) > 0$ s.t.

if $\|\dot{\mathcal{P}}\| < \eta_\varepsilon$ & $\|\dot{\mathcal{Q}}\| < \eta_\varepsilon$,

then $|S(f; \dot{\mathcal{P}}) - L| < \varepsilon/2$ and

$|S(f; \dot{\mathcal{Q}}) - L| < \varepsilon/2$.

$\therefore |S(f; \dot{\mathcal{P}}) - S(f; \dot{\mathcal{Q}})|$

$\leq |S(f; \dot{\mathcal{P}}) - L| + |L - S(f; \dot{\mathcal{Q}})|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. (Done)

($\Leftarrow$)

<u>Step 1</u> $\exists$ seq. (δ_n) with $0 < \delta_{n+1} \leq \delta_n$, $\forall n = 1, 2, 3, \cdots$

such that if $\|\dot{\mathcal{P}}\| < \delta_n$ & $\|\dot{Q}\| < \delta_n$,

then $|S(f, \dot{\mathcal{P}}) - S(f, \dot{Q})| < \frac{1}{n}$.

<u>Pf of Step 1</u>

By assumption, for $\varepsilon = \frac{1}{n} > 0$, $\exists\ \eta_{\frac{1}{n}} > 0$ s.t.

if $\|\dot{\mathcal{P}}\| < \eta_{\frac{1}{n}}$ & $\|\dot{Q}\| < \eta_{\frac{1}{n}}$,

then $|S(f, \dot{\mathcal{P}}) - S(f, \dot{Q})| < \frac{1}{n}$.

Let $\delta_n = \min\{\eta_1, \eta_{\frac{1}{2}}, \cdots, \eta_{\frac{1}{n}}\} > 0$, $\forall n = 1, 2, 3, \cdots$

then $\delta_{n+1} = \min\{\eta_1, \eta_{\frac{1}{2}}, \cdots, \eta_{\frac{1}{n}}, \eta_{\frac{1}{n+1}}\}$

$\leq \min\{\eta_1, \eta_{\frac{1}{2}}, \cdots, \eta_{\frac{1}{n}}\} = \delta_n$

And if $\|\dot{\mathcal{P}}\| < \delta_n$ & $\|\dot{Q}\| < \delta_n$

then $\|\dot{\mathcal{P}}\| < \eta_{\frac{1}{n}}$ & $\|\dot{Q}\| < \eta_{\frac{1}{n}}$

$\therefore\ |S(f, \dot{\mathcal{P}}) - S(f, \dot{Q})| < \frac{1}{n}$. ※

<u>Step 2</u> $\exists$ a seq. of tagged partition $\dot{Q}_n$ s.t.

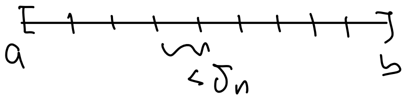
$\|\dot{Q}_n\| < \delta_n$ and (δ_n given in Step 1)

$\lim_{n \to \infty} S(f, \dot{Q}_n) = L$ exists.

Pf of Step 2:

For each $n = 1, 2, 3, \cdots$, choose any tagged partition $\dot{Q}_n$ with $\|\dot{Q}_n\| < \delta_n$

(Existence of $\dot{Q}_n$ is clear: for example, one may take uniform partition with any tags



)

Then, for $m \geq n \geq 1$, we have

$$\|\dot{Q}_n\| < \delta_n \quad \text{and} \quad \|\dot{Q}_m\| < \delta_m \leq \delta_n$$

$\therefore$ Step 1 $\Rightarrow$ $$|S(f, \dot{Q}_n) - S(f, \dot{Q}_m)| < \frac{1}{n} \quad \text{——} (*),$$

Hence $\forall \varepsilon > 0$, one can take any integer $n_0 > \frac{1}{\varepsilon}$ and conclude that $\forall\, m \geq n \geq n_0$,

$$|S(f, \dot{Q}_n) - S(f, \dot{Q}_m)| < \frac{1}{n_0} < \varepsilon$$

$\therefore$ $(S(f, \dot{Q}_n))$ is a Cauchy sequence.

By completeness of $\mathbb{R}$ (Thm 3.5.5 Cauchy Convergence Criterion),

$\lim_{n\to\infty} S(f, \dot{Q}_n)$ exists (let call it L)

※

Final Step : $f \in \mathcal{R}[a,b]$

Pf of Final Step

Using Step 2 and $(*)_1$, by taking $m \to \infty$, we have

$$\left| S(f, \dot{Q}_n) - L \right| \le \frac{1}{n}, \quad \forall n = 1, 2, 3, \cdots \quad \text{---} (*)_2$$

Now $\forall \varepsilon > 0$, k is an integer s.t. $k > \frac{2}{\varepsilon}$.

Then if $\dot{\mathcal{P}}$ satisfies $\|\dot{\mathcal{P}}\| < \delta_k$,

we have $\left| S(f, \dot{\mathcal{P}}) - S(f, \dot{Q}_k) \right| < \frac{1}{k}$ by Step 1

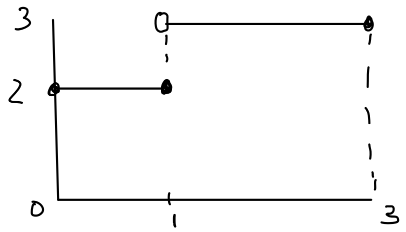
and hence

$$\begin{aligned} \left| S(f, \dot{\mathcal{P}}) - L \right| &\le \left| S(f, \dot{\mathcal{P}}) - S(f, \dot{Q}_k) \right| + \left| S(f, \dot{Q}_k) - L \right| \\ &< \frac{1}{k} + \frac{1}{k} \qquad (\text{by } (*)_2) \\ &= \frac{2}{k} < \varepsilon \end{aligned}$$

$\therefore \ f \in \mathcal{R}[a,b] \ \left(\& \int_a^b f = L \right)$ ⧣

Eg 7.2.2

(a) $g:[0,3]\to\mathbb{R}$ defined by $g(x)=\begin{cases} 3, & 1<x\le 3 \\ 2, & 0\le x\le 1\end{cases}$



is (Riemann) integrable, Eg 7.1.4(b)

In Eg 7.1.4(b), we proved that

if $\|\dot{\mathcal{P}}\|<\delta$, then

$$8-5\delta\le S(g,\dot{\mathcal{P}})\le 8+5\delta.$$

If $\dot{Q}$ is another one with $\|\dot{Q}\|<\delta$, we also have

$$8-5\delta\le S(g,\dot{Q})\le 8+5\delta.$$

Hence $|S(g,\dot{\mathcal{P}})-S(g,\dot{Q})|\le(8+5\delta)-(8-5\delta)=10\delta$

$\therefore\ \forall\,\varepsilon>0,\ \exists\,\eta_\varepsilon=\frac{\varepsilon}{20}>0$ s.t.

if $\|\dot{\mathcal{P}}\|<\eta_\varepsilon$ & $\|\dot{Q}\|<\eta_\varepsilon$,

then $|S(g,\dot{\mathcal{P}})-S(g,\dot{Q})|\le 10\cdot\frac{\varepsilon}{20}=\frac{\varepsilon}{2}<\varepsilon$.

$\therefore$ Cauchy Criterion is satisfied.

(b) Applying Cauchy Criterion to show a function is <u>not</u> integrable:

f is not integrable $\Leftrightarrow\ \exists\,\varepsilon_0>0$, s.t. $\forall\,\eta>0$,

$\exists\,\dot{\mathcal{P}},\dot{Q}$ with $\|\dot{\mathcal{P}}\|<\eta$ & $\|\dot{Q}\|<\eta$ s.t.

$$|S(f,\dot{\mathcal{P}})-S(f,\dot{Q})|\ge\varepsilon_0$$

Concrete example:

Dirichlet function $f(x) = \begin{cases} 1, & \text{if } x \text{ rational}, x \in [0,1] \\ 0, & \text{if } x \text{ irrational}, x \in [0,1]. \end{cases}$

(eg 5.1.6 (g))

Consider $\varepsilon_0 = \frac{1}{2} > 0$,

$\forall \eta > 0$, let $\dot{\mathcal{P}}$ = any partition s.t. $\|\dot{\mathcal{P}}\| < \eta$
with rational tags. (ie. all $t_i \in \mathbb{Q} \cap [0,1]$)

$\dot{\mathcal{Q}}$ = any partition s.t. $\|\dot{\mathcal{Q}}\| < \eta$
with irrational tags. (ie. all $t_i \in [0,1] \setminus \mathbb{Q}$)

Then $S(f, \dot{\mathcal{P}}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} (x_i - x_{i-1}) = 1$

$S(f, \dot{\mathcal{Q}}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) = 0$

$\Rightarrow \quad |S(f, \dot{\mathcal{P}}) - S(f, \dot{\mathcal{Q}})| = 1 \geq \varepsilon_0$

$\therefore$ f is not Riemann integrable.

Thm 7.2.3 (Squeeze Thm) Let $f : [a,b] \to \mathbb{R}$ $(a < b)$

Then $f \in \mathcal{R}[a,b] \Leftrightarrow \forall \varepsilon > 0$, $\exists$ functions α_ε and $\omega_\varepsilon \in \mathcal{R}[a,b]$

with $\alpha_\varepsilon(x) \leq f(x) \leq \omega_\varepsilon(x)$, $\forall x \in [a,b]$

such that $\int_a^b (\omega_\varepsilon - \alpha_\varepsilon) < \varepsilon$.

(Remark: We don't need to assume $\lim_{\varepsilon\to 0}\int_a^b \omega_\varepsilon$ or $\lim_{\varepsilon\to 0}\int_a^b \alpha_\varepsilon$ exist,

but of course their existence follows from Thm 7.1.5(c))

Pf: ($\Rightarrow$) If $f\in\mathcal{R}[a,b]$, take $\alpha_\varepsilon \equiv f \equiv \omega_\varepsilon$, $\forall \varepsilon>0$

Then $\int_a^b (f-f) = 0 < \varepsilon$

($\Leftarrow$) By assumption, α_ε & $\omega_\varepsilon \in \mathcal{R}[a,b]$.

Hence $\forall\varepsilon>0$, $\exists\, \delta_\varepsilon>0$ such that

if $\|\dot{\mathcal{P}}\|<\delta_\varepsilon$, then $\left|S(\alpha_\varepsilon,\dot{\mathcal{P}})-\int_a^b\alpha_\varepsilon\right|<\varepsilon$

and $\left|S(\omega_\varepsilon,\dot{\mathcal{P}})-\int_a^b\omega_\varepsilon\right|<\varepsilon$

(This $\delta_\varepsilon=\min\{\delta_\varepsilon',\delta_\varepsilon''\}$, where δ_ε' is for α_ε, δ_ε'' for ω_ε)

Therefore $\int_a^b\alpha_\varepsilon-\varepsilon<S(\alpha_\varepsilon,\dot{\mathcal{P}})$

and $S(\omega_\varepsilon,\dot{\mathcal{P}})<\int_a^b\omega_\varepsilon+\varepsilon$

Since $\alpha_\varepsilon(x)\le f(x)\le\omega_\varepsilon(x)$, $\forall x\in[a,b]$, we have

$$S(\alpha_\varepsilon,\dot{\mathcal{P}})\le S(f,\dot{\mathcal{P}})\le S(\omega_\varepsilon,\dot{\mathcal{P}}).$$

$\therefore$

$$\int_a^b\alpha_\varepsilon-\varepsilon<S(f,\dot{\mathcal{P}})<\int_a^b\omega_\varepsilon+\varepsilon$$

Therefore, if $\dot{\mathcal{P}}$ & $\dot{Q}$ are two tagged partitions with

$\|\dot{\mathcal{P}}\|<\delta_\varepsilon$ and $\|\dot{Q}\|<\delta_\varepsilon$, we have $\int_a^b \alpha_\varepsilon - \varepsilon < S(f, \dot{\mathcal{P}}) < \int_a^b \omega_\varepsilon + \varepsilon$

and $\int_a^b \alpha_\varepsilon - \varepsilon < S(f, \dot{Q}) < \int_a^b \omega_\varepsilon + \varepsilon$

$$\Rightarrow \quad |S(f, \dot{\mathcal{P}}) - S(f, \dot{Q})| < \left(\int_a^b \omega_\varepsilon + \varepsilon\right) - \left(\int_a^b \alpha_\varepsilon - \varepsilon\right)$$

$$= \int_a^b (\omega_\varepsilon - \alpha_\varepsilon) + 2\varepsilon$$

$$< \varepsilon + 2\varepsilon = 3\varepsilon .$$

Since $\varepsilon > 0$ is arbitrary, f satisfies Cauchy Criterion

$\therefore f \in \mathcal{R}[a,b]$ #

<u>Recall</u> (Def 5.4.9 of the Textbook)

A function $\varphi : [a,b] \to \mathbb{R}$ is a <u>step function</u>

if $\exists$ subintervals I_i (not necessary closed) with

$$\begin{cases} I_i \cap I_j = \phi \text{ for } i \neq j \text{ and} \\ [a,b] = \bigcup_{i=1}^{n} I_i , \end{cases}$$

such that $\varphi|_{I_i} = \text{const function on } I_i$,

i.e. $\varphi(x) = k_i , \forall x \in I_i$ (for some k_i)

Lemma 7.2.4 Let • J = subinterval of $[a,b]$,

• $c<d$ are endpoints of J

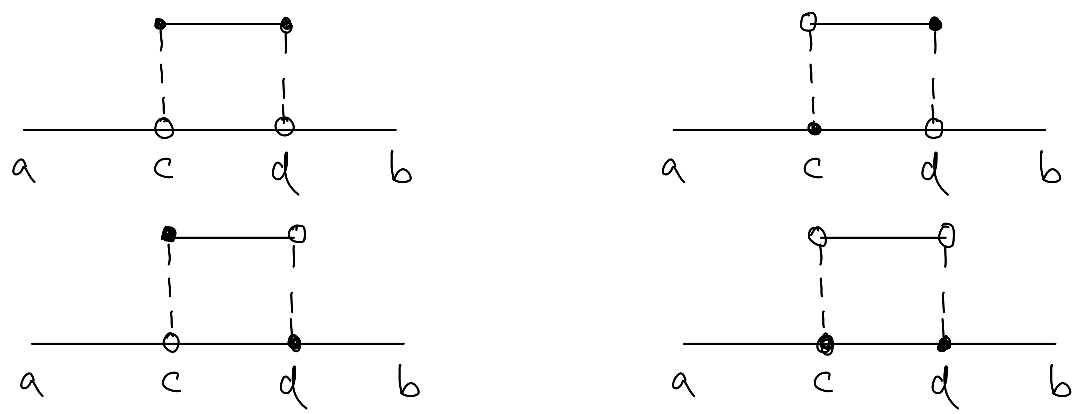
If
$$\varphi_J(x) = \begin{cases} 1 & \text{for } x\in J \\ 0 & \text{for } x\notin J \quad (x\in[a,b]) \end{cases}$$

then $\varphi_J \in \mathcal{R}[a,b]$ and $\int_a^b \varphi_J = d-c$.

Pf: There are 4 cases for J:

$$J = [c,d],\ (c,d],\ [c,d) \text{ and } (c,d)$$

and corresponding 4 cases of φ_J



All these 4 cases different from each others by a finitely many points (at most 2), therefore all 4 cases have the same integral by Thm 7.1.3

By Ex 7.1.13 (assigned in Homework 4), we have

$$\int_a^b \varphi_J = d-c \quad \text{for the case of } J=[c,d].$$

Hence $\int_a^b \varphi_J = d-c$ for all cases. ※

Thm 7.2.5 If $\varphi : [a,b] \to \mathbb{R}$ is a <u>step function</u>, then $\varphi \in \mathcal{R}[a,b]$

(i.e. step functions are Riemann integrable)

<u>Pf</u>: Assume $\varphi(x) = k_i$ for $x \in I_i$ $\left(I_i \cap I_j = \phi,\ \bigcup_{i=1}^{n} I_i = [a,b] \right)$

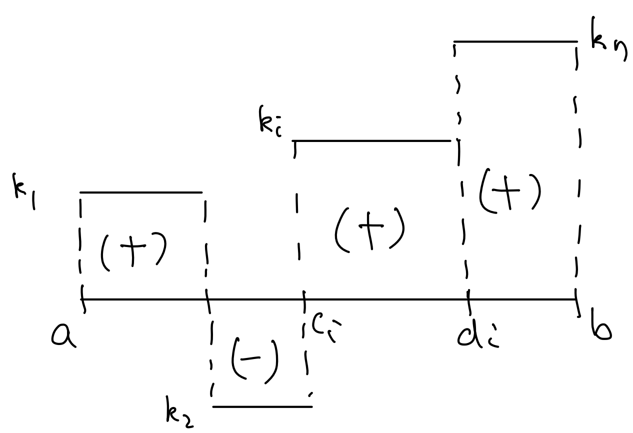
Then by using the notations in Lemma 7.2.4,

$$\varphi(x) = \sum_{i=1}^{n} k_i \varphi_{I_i}(x).$$

Since $\varphi_{I_i} \in \mathcal{R}[a,b]$, Thm 7.1.5 (a) & (b) $\Rightarrow$ $\varphi \in \mathcal{R}[a,b]$ $\divideontimes$

Moreover, if $I_i = [c_i, d_i]$, $i = 1, \cdots, n$, then Lemma 7.2.4 $\Rightarrow$

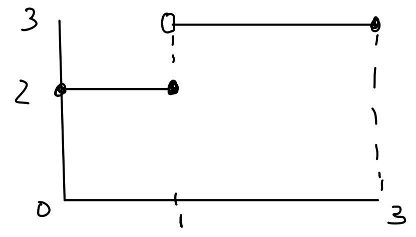
$$\int_a^b \varphi = \sum_{i=1}^{n} k_i \int_a^b \varphi_{I_i} = \sum_{i=1}^{n} k_i (d_i - c_i).$$



Eg 7.2.6

(a) (Eg 7.1.4 (b) again)

$g: [0,3] \to \mathbb{R}$ defined by $g(x) = \begin{cases} 3, & 1 < x \le 3 \\ 2, & 0 \le x \le 1 \end{cases}$



is a step function.

Clearly $g(x) = 2\varphi_{[0,1]}(x) + 3\varphi_{(1,3]}(x) \in \mathcal{R}[0,3]$

and $\int_a^b g = 2\int_a^b \varphi_{[0,1]} + 3\int_a^b \varphi_{(1,3]}$

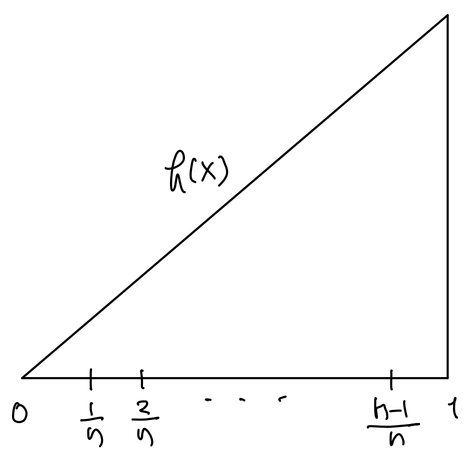
$= 2\cdot(1-0) + 3\cdot(3-1) = 8$.

(b) (eg 7.1.4 (c))

$h(x) = x$ on $[0,1]$.

Consider (uniform) partition

$\mathcal{P}_n = \{[\frac{k-1}{n}, \frac{k}{n}]\}_{k=1}^n$

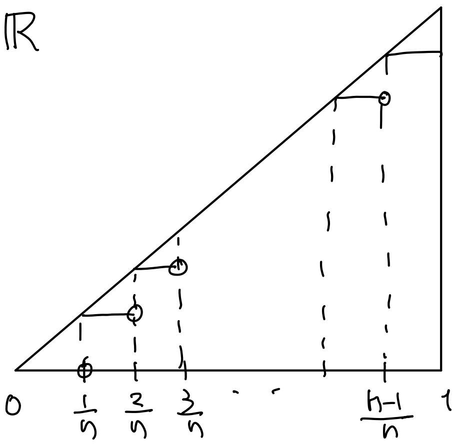


Define step functions $\alpha_n : [0,1] \to \mathbb{R}$

by (for $k=1,\dots,n$)

$\alpha_n(x) = \frac{k-1}{n}$ for $x \in \left[\frac{k-1}{n}, \frac{k}{n}\right)$,

and $\alpha_n(1) = \frac{n-1}{n}$

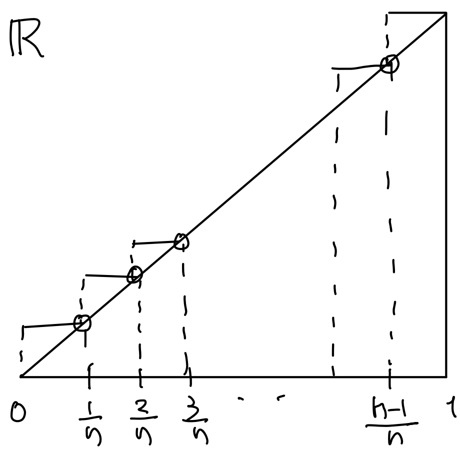


Define step functions $\omega_n : [0,1] \to \mathbb{R}$

by (for $k=1,\dots,n$)

$\omega_n(x) = \frac{k}{n}$ for $x \in \left[\frac{k-1}{n}, \frac{k}{n}\right)$,

and $\omega_n(1) = 1$



Then clearly $\alpha_n(x) \le h(x) \le \omega_n(x) \quad \forall x \in [0,1]$,

as $\alpha_n(x) = \min\limits_{\left[\frac{k-1}{n}, \frac{k}{n}\right)} h(x)$ & $\omega_n(x) = \sup\limits_{\left[\frac{k-1}{n}, \frac{k}{n}\right)} h(x)$

for $x \in \left[\frac{k-1}{n}, \frac{k}{n}\right)$ $\left(\left[\frac{n-1}{n}, 1\right] \text{ for } k=n \right)$

By Thm 7.2.5

$$\int_0^1 \alpha_n = 0 \cdot \frac{1}{n} + \frac{1}{n} \cdot \frac{1}{n} + \frac{2}{n} \cdot \frac{1}{n} \cdots + \frac{(n-1)}{n} \cdot \frac{1}{n}$$

$$= \frac{1}{n^2}(1 + \cdots + (n-1)) = \frac{1}{n^2} \cdot \frac{(n-1)n}{2}$$

$$= \frac{1}{2}\left(1 - \frac{1}{n}\right)$$

$$\int_0^1 \omega_n = \frac{1}{n}\cdot\frac{1}{n} + \frac{2}{n}\cdot\frac{1}{n} + \frac{3}{n}\cdot\frac{1}{n} \cdots + \frac{n}{n}\cdot\frac{1}{n}$$

$$= \frac{1}{n^2}(1+\cdots+n) = \frac{1}{n^2}\,\frac{n(n+1)}{2}$$

$$= \frac{1}{2}\left(1+\frac{1}{n}\right)$$

Hence (by Thm 7.1.5)

$$\int_0^1 (\omega_n - \alpha_n) = \frac{1}{n}$$

$\therefore$ $\forall \varepsilon > 0$, choose n_ε s.t. $\frac{1}{n_\varepsilon} < \varepsilon$, then

$$\alpha_{n_\varepsilon}(x) \le h(x) \le \omega_{n_\varepsilon}(x) \quad \text{s.t.}$$

$$\int_0^1 (\omega_{n_\varepsilon} - \alpha_{n_\varepsilon}) = \frac{1}{n_\varepsilon} < \varepsilon \quad .$$

By Squeeze Thm (7.2.3), $h(x) = x \in \mathcal{R}[0,1]$.

Furthermore, by Thm 7.1.5

$$\frac{1}{2}\left(1-\frac{1}{n_\varepsilon}\right) = \int_0^1 \alpha_{n_\varepsilon} \le \int_0^1 h \le \int_0^1 \omega_{n_\varepsilon} = \frac{1}{2}\left(1+\frac{1}{n_\varepsilon}\right)$$

letting $\varepsilon \to 0$, we have $n_\varepsilon \to \infty$, and hence

$$\int_0^1 h = \frac{1}{2} \, .$$

Thm 7.2.7 If $f:[a,b]\to\mathbb{R}$ is continuous, then $f\in\mathcal{R}[a,b]$. $(-\infty<a<b<+\infty)$

(Continuous functions on closed & bounded interval are Riemann integrable)

Pf: By Thm 5.4.3, cont. functions on closed & bounded interval are uniformly continuous.

$\therefore$ $\forall \varepsilon>0$, $\exists\, \delta_\varepsilon>0$ (indep. of points) such that

if $|x-y|<\delta_\varepsilon$ $(x,y\in[a,b])$,

then $$|f(x)-f(y)|<\frac{\varepsilon}{b-a}$$

Then take any partition $\mathcal{P}=\{[x_{i-1},x_i]\}_{i=1}^{n}$ such that

$\|\mathcal{P}\|<\delta_\varepsilon$. (such $\mathcal{P}$ always exists)

Since f is continuous

$\exists\, t_i'\in[x_{i-1},x_i]$ such that $f(t_i')=\min\limits_{[x_{i-1},x_i]} f(x)$ and

$\exists\, t_i''\in[x_{i-1},x_i]$ such that $f(t_i'')=\max\limits_{[x_{i-1},x_i]} f(x)$

Define step functions

$$\alpha_\varepsilon(x)=\begin{cases} f(t_i') & \text{for } x\in[x_{i-1},x_i) \text{ for } i\neq n \\ f(t_n') & \text{for } x\in[x_{n-1},x_n]. \end{cases}$$

and

$$\omega_\varepsilon(x)=\begin{cases} f(t_i'') & \text{for } x\in[x_{i-1},x_i) \text{ for } i\neq n \\ f(t_n'') & \text{for } x\in[x_{n-1},x_n]. \end{cases}$$

Then $\quad \alpha_\varepsilon(x) \leq f(x) \leq \omega_\varepsilon(x) \qquad \forall\, x \in [a,b].$

Moreover,

$$\int_a^b (\omega_\varepsilon - \alpha_\varepsilon) = \sum_{i=1}^{n} \left(f(t_i'') - f(t_i')\right)(x_i - x_{i-1})$$

$$< \sum_{i=1}^{n} \left(\frac{\varepsilon}{b-a}\right)(x_i - x_{i-1})$$

since $t_i'', t_i' \in [x_{i-1}, x_i] \Rightarrow |t_i'' - t_i'| \leq |x_i - x_{i-1}| \leq \|P\| < \delta_\varepsilon$

$$\therefore \quad \int_a^b (\omega_\varepsilon - \alpha_\varepsilon) < \frac{\varepsilon}{b-a} \cdot (b-a) = \varepsilon$$

Hence Squeeze Thm (7.2.3) $\Rightarrow f \in \mathcal{R}[a,b]$ ※

$\boxed{\text{\underline{Thm 7.2.8} If } f:[a,b] \to \mathbb{R} \text{ is \underline{monotone} on } [a,b], \text{ then } f \in \mathcal{R}[a,b]}$

$(-\infty < a < b < +\infty)$

Pf: Suppose f is <u>increasing</u> (<u>decreasing</u> are similar)

Take uniform partition $\mathcal{P} = \{[x_{i-1}, x_i]\}_{i=1}^{n}$ such that

$$x_i - x_{i-1} = \frac{b-a}{n} \quad \forall\, i = 1, 2, \cdots, n \quad (\text{with } x_0 = a)$$

Then $\quad f(x_{i-1}) \leq f(x) \leq f(x_i), \ \forall x \in [x_{i-1}, x_i] \ (\forall\, i = 1, \cdots, n)$

Define step functions

$$\alpha_n(x) = \begin{cases} f(x_{i-1}), & x \in [x_{i-1}, x_i) \\ f(x_{n-1}), & x \in [x_{n-1}, x_n] \end{cases}$$

and $$\omega_n(x) = \begin{cases} f(x_i), & x \in [x_{i-1}, x_i) \\ f(x_n), & x \in [x_{n-1}, x_n] \end{cases}$$

Then $$\alpha_n(x) \le f(x) \le \omega_n(x), \quad \forall x \in [a,b]$$

and $$\int_a^b \alpha_n = \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1})$$

$$= \frac{b-a}{n}\left[f(x_0) + f(x_1) + \cdots + f(x_{n-1})\right]$$

$$\int_a^b \omega_n = \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})$$

$$= \frac{b-a}{n}\left[f(x_1) + f(x_2) + \cdots + f(x_n)\right]$$

$$\therefore \int_a^b (\omega_n - \alpha_n) = \frac{b-a}{n}\left[f(x_n) - f(x_0)\right] = \frac{(b-a)(f(b)-f(a))}{n}$$

Hence $\forall \varepsilon > 0$, $\exists\, n_\varepsilon > \dfrac{(b-a)(f(b)-f(a))}{\varepsilon}$ s.t.

$$\alpha_{n_\varepsilon}(x) \le f(x) \le \omega_{n_\varepsilon}(x), \ \forall x \in [a,b] \ \& \ \int_a^b (\omega_{n_\varepsilon} - \alpha_{n_\varepsilon}) < \varepsilon$$

$\therefore f \in \mathcal{R}[a,b]$ by Squeeze Thm 7.2.3 ※