Unit Quaternions and Rotations in $\mathbb{R}^3$

<u>Thm</u> (i) Let r be a <u>unit</u> quaternion. Let R be a transformation (of $\mathbb{R}^3$) defined by

$$Rq = rqr^* \qquad \left(\begin{aligned} R: \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ q &\longmapsto rqr^* \end{aligned} \right)$$

where q is a pure quaternion.

Then R is a <u>rotation</u> of a 3-dim'l space of pure quaternions about an axis passing thro. the origin.

(ii) Specifically, if the polar form of r is

$$r = \cos\theta + u\sin\theta,$$

where u is a <u>pure unit quaternion</u>.

Then Rq is the pure quaternion obtained by rotating q about u by the <u>angle 2θ</u>.

(iii) Every rotation of 3-dim'l space (about an axis passing thro. the origin) can be expressed in this way.

Pf of (ii) :

Case 1 : $q = u$ ($q = \lambda u$, $\lambda \in \mathbb{R}$)

Then $Ru = rur^*$

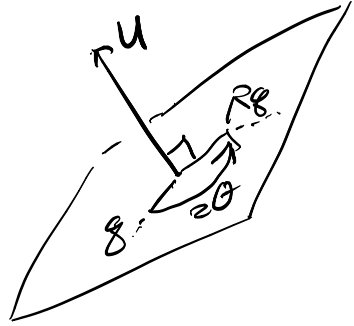
$$= (\cos\theta + u\sin\theta)u(\cos\theta - u\sin\theta)$$
$$= (u\cos\theta + u^2\sin\theta)(\cos\theta - u\sin\theta)$$
$$= (u\cos\theta - \sin\theta)(\cos\theta - u\sin\theta)$$
$$= u\cos^2\theta - \sin\theta\cos\theta - u^2\cos\theta\sin\theta + u\sin^2\theta$$
$$= u(\cos^2\theta + \sin^2\theta) - \sin\theta\cos\theta + \cos\theta\sin\theta$$
$$= u$$

$\therefore$ Ru is pure quaternion

and u is fixed point of R.

And immediately, we have $R(\lambda u) = \lambda u$

$\therefore$ the axis in the direction of u is fixed by R.



Case 2 q is perpendicular to u.

In this case,

$Rq = rqr^*$

$$= (\cos\theta + u\sin\theta)\, q\, (\cos\theta - u\sin\theta)$$

$$= (q\cos\theta + uq\sin\theta)(\cos\theta - u\sin\theta)$$

$$= q\cos^2\theta + uq\sin\theta\cos\theta - qu\cos\theta\sin\theta$$
$$- uqu\sin^2\theta$$

Since u, q are pure quaternions & $q \perp u$,

then $qu = -uq$ (ex6. of HW2)

and hence $uqu = u(qu) = u(-uq)$

$$= -u^2 q$$
$$= q.$$

Therefore

$$Rq = q\cos^2\theta + (uq - qu)\cos\theta\sin\theta - uqu\sin^2\theta$$

$$= q\cos^2\theta + (uq + uq)\cos\theta\sin\theta - q\sin^2\theta$$

$$= q(\cos^2\theta - \sin^2\theta) + (2\sin\theta\cos\theta)\, uq$$

$$= (\cos 2\theta)\, q + (\sin 2\theta)\, uq.$$

Note that u, q are pure quaternions,

$$uq = -u \cdot q + u \times q$$

$$= u \times q \quad (\text{since } u \perp q \Leftrightarrow u \cdot q = 0)$$

$\therefore$ uq is also a pure quaternion.

$\therefore$ $Rq = (\cos 2\theta) q + (\sin 2\theta) uq \in \mathbb{R}^3$ (pure quaternion.)

Also • $|uq| = |u||q| = |q|$ (Ex!), and

• $(uq)q = (-qu)q$ (by $uq = -qu$)

$$= -q(uq)$$

by ex6. of HW2 $\Rightarrow$ $uq \perp q$, and

• $u(uq) = u(-qu) = -(uq)u$

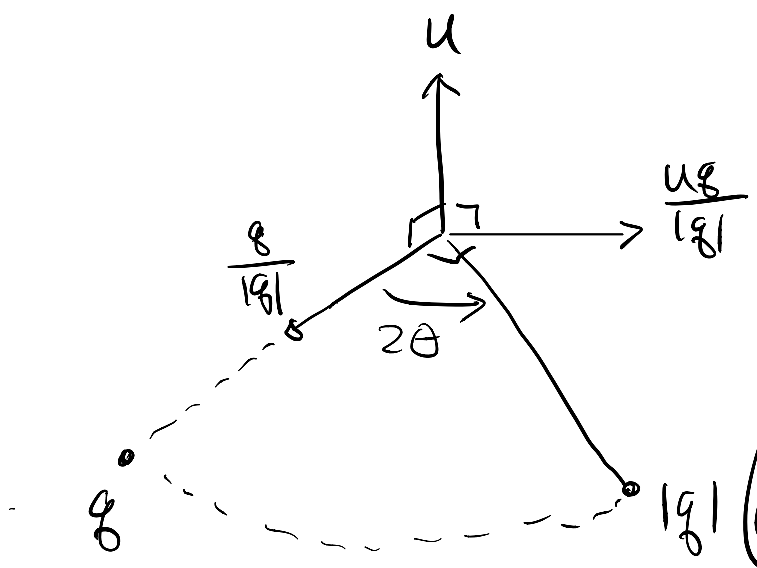
$\Rightarrow u \perp uq$ (ex6. of HW2)

Hence $\left\{ \frac{q}{|q|}, \frac{uq}{|q|} \right\}$ is an orthonormal basis

for the plane perpendicular to u.

$$\therefore \quad Rq = (\cos 2\theta)\, q + \sin 2\theta \; uq$$

is the rotation of q thro. an angle of 2θ about the axis in the direction of u.



$$|q|\left((\cos 2\theta)\frac{q}{|q|} + (\sin 2\theta)\frac{uq}{|q|}\right)$$

$$= (\cos 2\theta)\, q + (\sin 2\theta)\, uq$$

$$= Rq.$$

<u>Case 3</u> : <u>General pure quaternions</u>

Note that R is a linear transformation

$$\begin{cases} R(q_1 + q_2) = Rq_1 + Rq_2 \\ R(\lambda q) = \lambda R q \end{cases}, \quad \left(\begin{array}{c} \forall \text{ pure quaternions } \\ q, q_1, q_2 \in \mathbb{R}^3 \\ \lambda \in \mathbb{R} \end{array} \right)$$

Similarly, a rotation in $\mathbb{R}^3$ is also linear.

Denote $\mathcal{O}$ = the rotation thro. an angle of 2θ about the axis of u.

Then any pure quaternion p can be written as

$$p = \lambda u + q$$

where $\lambda \in \mathbb{R}$ and $q \perp u$.

$$\Rightarrow \quad Rp = R(\lambda u + q)$$
$$= \lambda Ru + Rq$$
$$= \lambda \mathcal{O}u + \mathcal{O}q$$
$$= \mathcal{O}(\lambda u + q) = \mathcal{O}p.$$

$\therefore \; R \equiv \mathcal{O}$.

(Pf of (i) & (iii) are easy (Ex!))

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Remarks:

- $(-r)\,q\,(-r)^* = r q r^*$ (r = unit quaternion)

  Hence $\pm r \longmapsto$ the same rotation in $\mathbb{R}^3$.

- Translation : $Tq = q + b$, where p is pure quaternion.

Ch 18 & 19 3-Dimensional Euclidean and Hyperbolic Geometry (Solid Geometry)

Euclidean Solid Geometry

Def: Let $\mathbb{V} = \{ v = xi + yj + zk : x, y, z \in \mathbb{R} \} \ (\neq \phi)$

be the set of pure quaternions and

$$\mathbb{IR} = \left\{ T: \mathbb{V} \to \mathbb{V} : Tv = rvr^* + b \text{ for some unit quaternion } r \text{ and pure quaternion } b. \right\}$$

be a set of transformations (Euclidean transformations) of $\mathbb{V}$

The pair $(\mathbb{V}, \mathbb{IR})$ models Euclidean Solid Geometry.

Check that this is well-defined. i.e. elements in $\mathbb{IR}$ are really invertible transformations on $\mathbb{V}$, and $\mathbb{IR}$ satisfies the 3 requirments.

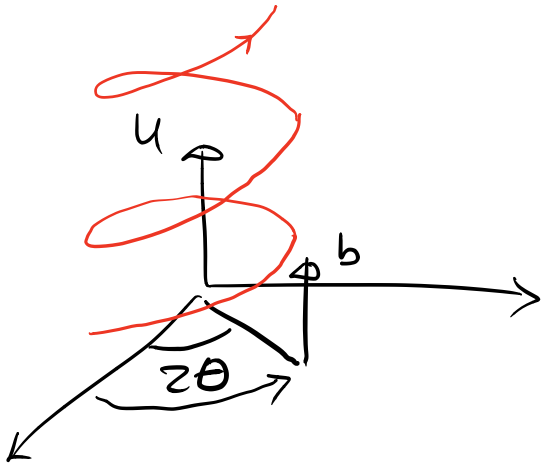
Screw motions

If $r = \cos\theta + u\sin\theta$, $\quad u$ = pure unit quaternion

& b parallel to u

then $Tv = rvr^* + b$

is called a screw motion



Thm: Every Euclidean transformation is a screw motion (but centered at different point)

Lemma 1: Every Euclidean transformation with a fixed point is a rotation.

Pf: (i) If 0 is a fixed point. Then

$$0 = T0 = r0r^* + b = b$$

$$\Rightarrow b = 0 \ \& \ Tv = rvr^* \text{ is a rotation.}$$

(ii) If q is a fixed point. Let S be a Euclidean transformation such that $Sq = 0$

(for instance : $Sv = v - q$, ie. $r = 1$).

Then STS^{-1} has 0 as fixed point :

$$STS^{-1}(0) = STq = Sq = 0 .$$

$\Rightarrow$ STS^{-1} is a rotation.

by (i) $\Rightarrow$ T is a rotation about an axis passing thro. q.

$(Tv = r(v-q)r^* + q.)$ ※

<u>Lemma 2</u> : Let $Tv = rvr^* + b \in \mathbb{R}$

& $r = \cos\theta + u\sin\theta$, $\theta \in \mathbb{R}$

u = unit pure quaternion.

If u and b are perpendicular, then T is a rotation about an axis <u>parallel</u> to u.

<u>Pf</u> : <u>Step 1</u> : $v_0 = \dfrac{1}{2\sin\theta} r^* u b$ is pure quaternion

<u>Pf of step 1</u> : Since u, b pure & $u \perp b$,

we have $ub = -u \cdot b + u \times q = u \times q$

$\therefore$ ub is pure quaternion.

Then $r^* ub = (\cos\theta + u\sin\theta)^* ub$

$= (\cos\theta - u\sin\theta) ub$

$= (\cos\theta) ub - u(ub)\sin\theta$

$= (\cos\theta) ub + b\sin\theta$ is pure quaternion

Hence $v_0 = \frac{1}{2\sin\theta} r^* ub$ is also pure quaternion.

<u>Step 2</u>: (i) $bu = -ub$ (Ex 6 of HW2)

(ii) $ur = ru$ (note: r <u>not</u> pure)

(iii) $br^* = rb$

<u>Pf of Step 2 (ii)</u> $u(\cos\theta + u\sin\theta) = u\cos\theta - \sin\theta$

$(\cos\theta + u\sin\theta) u = u\cos\theta + u^2\sin\theta$

$= u\cos\theta - \sin\theta$.

(iii) $br^* = b(\cos\theta + u\sin\theta)^* = b(\cos\theta - u\sin\theta)$

$= b\cos\theta - bu\sin\theta$

$= b\cos\theta + ub\sin\theta$ (by (i))

$= (\cos\theta + u\sin\theta) b$

$= rb$. ※

Step 3: v_0 is a fixed point of T

(and hence T is a rotation, by lemma 1)

Pf of Step 3:
$$T v_0 = r v_0 r^* + b$$
$$= r\left(\frac{1}{2\sin\theta} r^* u b\right) r^* + b$$
$$= \frac{1}{2\sin\theta} r r^* u b r^* + b$$
$(|r|^2 = rr^* = 1)$
$$= \frac{1}{2\sin\theta} u b r^* + b$$
(by (iii) of step 2)
$$= \frac{1}{2\sin\theta} u r b + b$$
$$= \frac{1}{2\sin\theta}\left[u(\cos\theta + u\sin\theta) + 2\sin\theta\right] b$$
$$= \frac{1}{2\sin\theta}\left[u\cos\theta - \sin\theta + 2\sin\theta\right] b$$
$$= \frac{1}{2\sin\theta}(u\cos\theta + \sin\theta) b$$
$$= \frac{1}{2\sin\theta}(u\cos\theta - u^2\sin\theta) b$$
$$= \frac{1}{2\sin\theta}(\cos\theta - u\sin\theta) u b$$
$$= \frac{1}{2\sin\theta} r^* u b = v_0$$

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Final Step: Rotation axis parallel to u.

Pf: Need to show that

$$v_0 + tu \quad \longleftarrow \text{(axis of } u\text{)}$$

are fixed points of T, $\forall t \in (-\infty, \infty)$

To see this:

$$\begin{aligned}
T(v_0 + tu) &= r(v_0 + tu)r^* + b \\
&= rv_0r^* + t\,rur^* + b \\
&= (rv_0r^* + b) + t\,rur^* \\
&= v_0 + t\,urr^* \quad \text{(Step 3 \& (ii) of Step 2)} \\
&= v_0 + tu
\end{aligned}$$

$(rr^* = 1)$

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Proof of the Thm:

Let $Tv = rvr^* + b$, $\quad r = \cos\theta + u\sin\theta$
$\quad b =$ pure quaternion.

Decompose $b = b_1 + b_2$ such that

$$b_1 \perp u, \quad b_2 \,//\, u$$

Then
$$\begin{aligned}
Tv &= rvr^* + b \\
&= (rvr^* + b_1) + b_2
\end{aligned}$$

rotation with axis parallel to u (by lemma 2)

b_2 translation parallel to u.

Hence T is a screw motion by definition. #