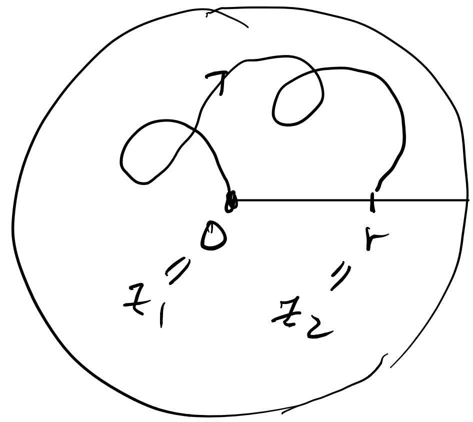
Pf of the Thm (Cont'd)

We may assume $z_1 = 0$ and $z_2 = r \in (0,1)$ (by a transformation)



Let γ be a (smooth) curve joining 0 and r with parametrization

$$z(t) = x(t) + i\,y(t), \quad a \le t \le b$$

with

$$\begin{cases} 0 = z_1 = z(a) = x(a) + i\,y(a) \\ r = z_2 = z(b) = x(b) + i\,y(b) \end{cases}$$

$$\Rightarrow \begin{cases} x(a) = y(a) = y(b) = 0 \\ x(b) = r \end{cases}$$

Then

$$\ell(\gamma) = 2\int_a^b \frac{|z'(t)|}{1 - |z(t)|^2}\,dt$$

$$= 2\int_a^b \frac{\sqrt{(x'(t))^2 + (y'(t))^2}}{1 - (x(t))^2 - (y(t))^2}\,dt$$

$$\geq 2\int_a^b \frac{\sqrt{(x'(t))^2}}{1-x(t)^2}\,dt$$

$$= 2\int_a^b \frac{|x'(t)|}{1-x(t)^2}\,dt$$

$$\geq 2\int_a^b \frac{x'(t)}{1-x(t)^2}\,dt$$

$$= 2\int_{x(a)}^{x(b)} \frac{ds}{1-s^2}$$ where $s = x(t)$

$\Rightarrow ds = x'(t)dt$

$t = a \to s = x(a)$

$t = b \to s = x(b)$

$$= 2\int_0^r \frac{ds}{1-s^2}$$

$$= \ln\frac{1+r}{1-r}$$

$$= d(0,r) = d(z_1, z_2)$$

$\therefore\ \ell(\gamma) \geq d(z_1,z_2) = \ell\left(\begin{array}{l}\text{hyperbolic straight line}\\ \text{segment joining } z_1 \,\&\, z_2\end{array}\right)$

#

<u>Note</u>: If fact, if $\ell(\gamma) = d(0,r)$

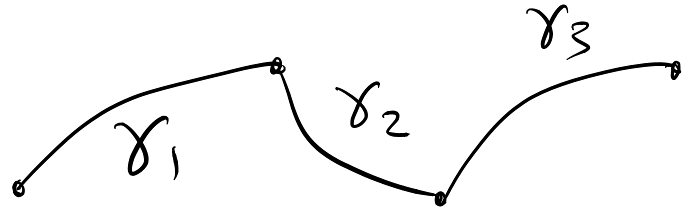
then $y' \equiv 0$ and $y \equiv 0$

and $x' = |x'| > 0$

$\Rightarrow$ after transformation, γ = x-axis between 0 and r (in the increasing direction)

$\Rightarrow$ γ = hyperbolic straight line segment joining z_1 & z_2

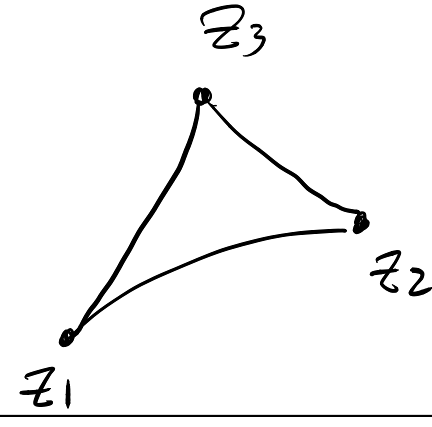
<u>Remark</u>: The Thm (& the Note) is actually true for piecewise smooth curves:



<u>Corollary</u> (<u>Triangle Inequality</u>)

For any 3 points z_1, z_2 & z_3 in the hyperbolic plane,

$$d(z_1, z_3) \leq d(z_1, z_2) + d(z_2, z_3)$$



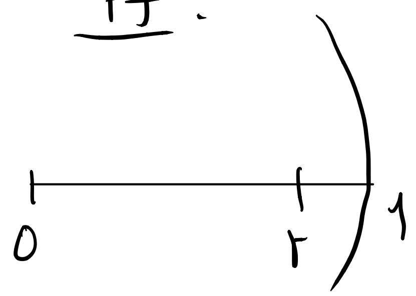
(Pf: Ex!)

Remark : The distance formula

$\Rightarrow$ Postulates 2 & 3 of Euclidean geometry also hold in hyperbolic geometry.

Postulate 2 : A line can be produced indefinitely in either direction.

Pf :



Since $\lim_{r \to 1} d(0, r) = \lim_{r \to 1} \ln \frac{1+r}{1-r} = +\infty$

$\forall N > 0, \exists\ 1 > r_1 > r$ s.t.

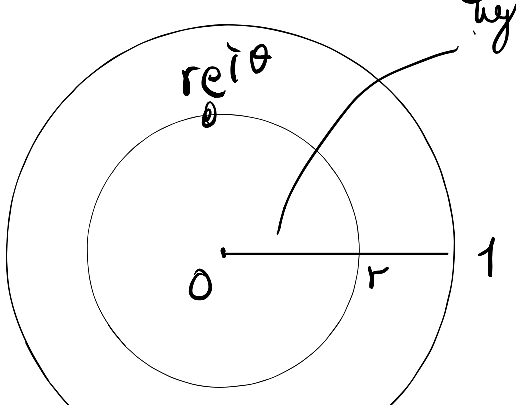
$$\ln \frac{1+r_1}{1-r_1} > \ln \frac{1+r}{1-r} + N$$

i.e. $\quad d(0, r_1) > d(0, r) + N$

$\therefore$ hyperbolic straight line segment can be produced indefinitely (i.e. longer than any prescribed length)

Postulate 3: A circle can be described with any center and radius.

Pf: Use a transformation, we only need to consider



Note that

$d(0, re^{i\theta})$

$= d(0,r) = \ln\frac{1+r}{1-r}$

$\therefore$ hyperbolic distance from a point $z = re^{i\theta}$ to the center 0 is a constant

$$d(0, re^{i\theta}) = \ln\frac{1+r}{1-r}$$

depending only on $r \in (0,1)$.

Hence $R = d(0, re^{i\theta}) = \ln\frac{1+r}{1-r}$

is the (hyperbolic) radius of the hyperbolic circle

And for any $R > 0$, we can solve $R = \ln\frac{1+r}{1-r}$

to find $r = \frac{e^R - 1}{e^R + 1}$ (Check), $r \in (0,1)$.

Then the Euclidean circle centered at O with Euclidean radius $r = \frac{e^R-1}{e^R+1}$ is the required hyperbolic circle centered as O with hyperbolic radius R. ✱

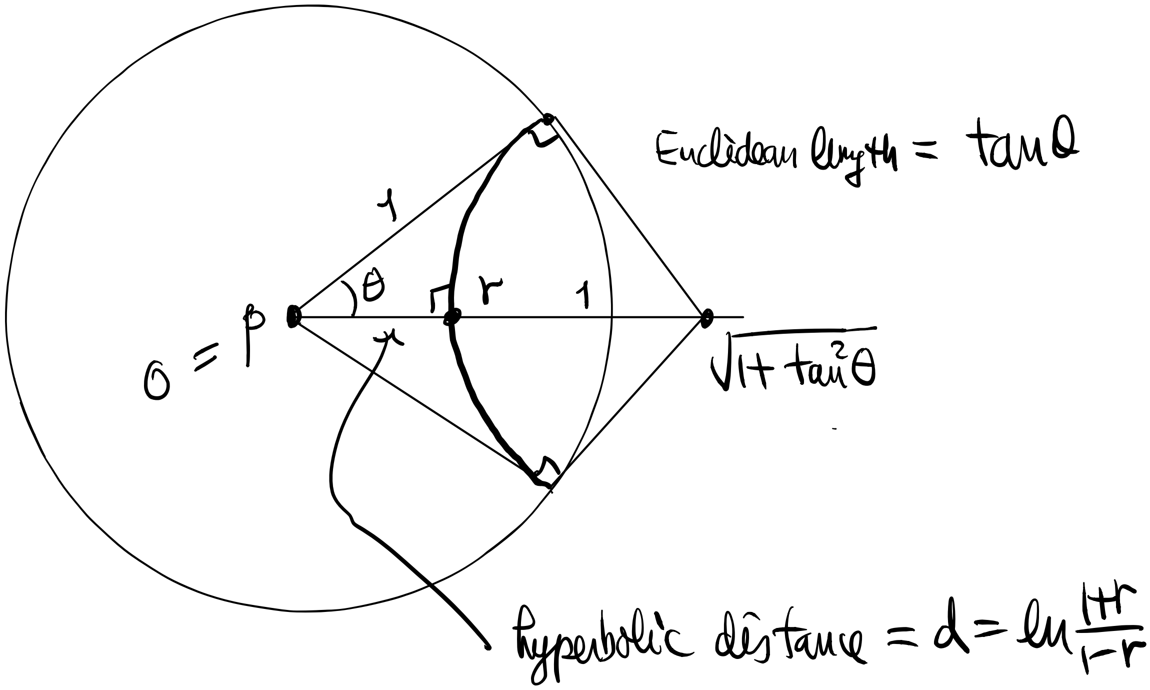
Conclusion: The hyperbolic geometry is a non-Euclidean geometry in the strict sense.

Famula of Lobatchevsky

Thm: Let the point p be given by the hyperbolic distance d from a hyperbolic straight line. Let θ be the angle of parallelism of p with respect to this line. Then

$$\boxed{e^{-d} = \tan\frac{\theta}{2}}$$

Pf:



After a transformation, we may assume $p = 0$ and the perpendicular from p to the hyperbolic straight line is the x-axis.

Then the point r as in the figure is given by

$$d = \ln \frac{1+r}{1-r}$$

and

$$r = \sqrt{1+\tan^2\theta} - \tan\theta$$

$$= \frac{1}{\cos\theta} - \tan\theta$$

$$= \frac{1-\sin\theta}{\cos\theta}$$

$$\Rightarrow \quad e^{-d} = \frac{1-r}{1+r} = \frac{1-\frac{1-\sin\theta}{\cos\theta}}{1+\frac{1-\sin\theta}{\cos\theta}} = \frac{\cos\theta+\sin\theta-1}{\cos\theta-\sin\theta+1}$$

$$= \frac{\left(\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\right)+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}-1}{\left(\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\right)-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}+1}$$

$$= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}-2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}$$

$\#$

The Upper Half Plane Model

<u>Def</u>: The <u>Upper half plane</u> is the subset

$$\mathbb{U} = \{ z : \operatorname{Im} z > 0 \} \subset \mathbb{C}$$

Let $\overline{\mathbb{H}}$ be the group of transformations (of $\mathbb{U}$) of the form

$$\left\{ w = Tz = \frac{az+b}{cz+d} \,, \quad \begin{matrix} a,b,c,d \in \mathbb{R} \\ \& \ ad-bc>0 \end{matrix} \right\}$$

The pair $(\mathbb{U}, \overline{\mathbb{H}})$ models hyperbolic geometry.

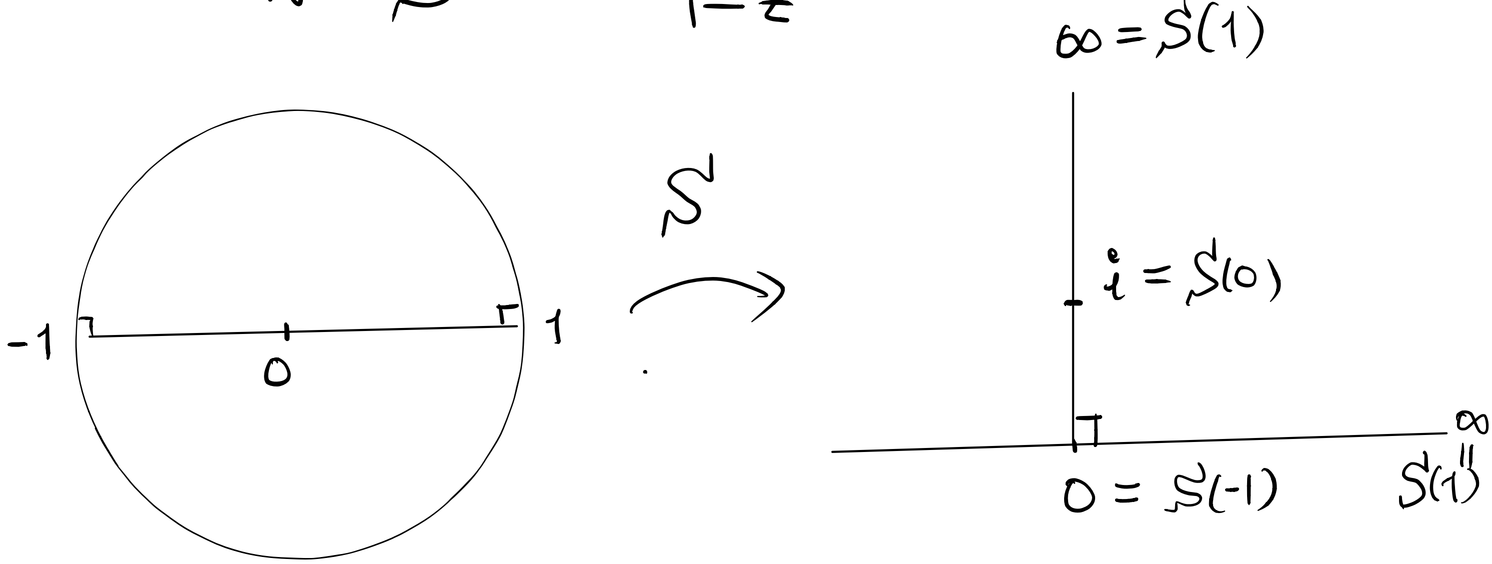
<u>Remark</u>: Both $(\mathbb{D}, \mathbb{H})$ and $(\mathbb{U}, \overline{\mathbb{H}})$ are models of the same abstract geometry, namely the hyperbolic geometry.

Distance in the upper half plane model

$$\boxed{\ell(\gamma) = \int_a^b \frac{|z'(t)|}{y(t)}\, dt \quad \text{for } \gamma : z(t) = x(t) + iy(t)}$$

Pf: Consider the transformation

$$w = Sz = i\,\frac{1+z}{1-z}$$



$S(-1) = 0,\quad S(0) = i,\quad S(1) = \infty$

Hence • $S : \mathbb{D} \to \mathbb{U}$

• $S^{-1}w = \dfrac{iw+1}{iw-1}$ (check!)

$$\boxed{\begin{array}{l} \bullet\ Tz = e^{i\theta}\dfrac{z-z_0}{1-\bar{z}_0 z},\quad z_0\in\mathbb{D},\ \theta\in\mathbb{R} \\ \qquad\qquad\qquad\qquad (\text{i.e. } T\in\mathbb{H}) \\ \Leftrightarrow S\circ T\circ S^{-1} \text{ is a transformation in } \overline{\mathbb{H}} \end{array}}$$

(Ex!)

$$\begin{tikzcd} \mathbb{D} \arrow[r, "T"] \arrow[d, "S"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{D} \arrow[d, "S"] \\ \mathbb{U} \arrow[r, "S\circ T\circ S^{-1}"] & \mathbb{U} \end{tikzcd}$$

$$T\in\mathbb{H} \;\Updownarrow\; S\circ T\circ S^{-1} \in \overline{\mathbb{H}}$$

$\therefore$ S is an isomorphism of the disk and upper half-plane models.

Now let $\gamma: z(t) = x(t) + i y(t)$, $a \le t \le b$ be a smooth curve in the upper half plane (i.e. $y(t) > 0$)

Then $\hat{\gamma}: \hat{z}(t) = S^{-1}(z(t))$ is a smooth curve in $\mathbb{D}$.

$$= \frac{i z(t) + 1}{i z(t) - 1}$$

$$\Rightarrow |\hat{z}'(t)| = \frac{|(i z(t) - 1)(i z'(t)) - (i z(t) + 1) i z'(t)|}{|i z(t) - 1|^2}$$

$$= \frac{2|z'(t)|}{|i z(t) - 1|^2}$$

$$\Rightarrow \underset{\substack{\uparrow \\ \text{upper half plane model}}}{l(\gamma)} = \underset{\substack{\uparrow \\ \text{disk model}}}{l(\hat{\gamma})} = 2\int_a^b \frac{|\hat{z}'|}{1 - |\hat{z}|^2} dt$$

$$= 2\int_a^b \frac{\frac{2|z'|}{|iz - 1|^2}}{1 - \left|\frac{iz+1}{iz-1}\right|^2} dt$$

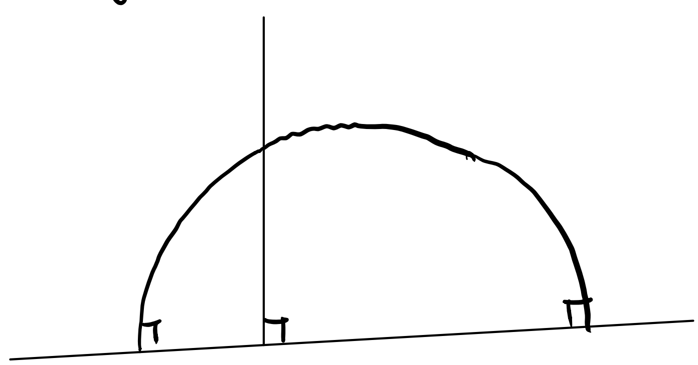
$$= \int_a^b \frac{4|z'|}{|iz-1|^2 - |iz+1|^2}\, dt$$

$$= \int_a^b \frac{4|z'|}{|ix-y-1|^2 - |ix-y+1|^2}\, dt$$

$$= \int_a^b \frac{4|z'|}{[(1+y)^2+x^2] - [(1-y)^2+x^2]}\, dt$$

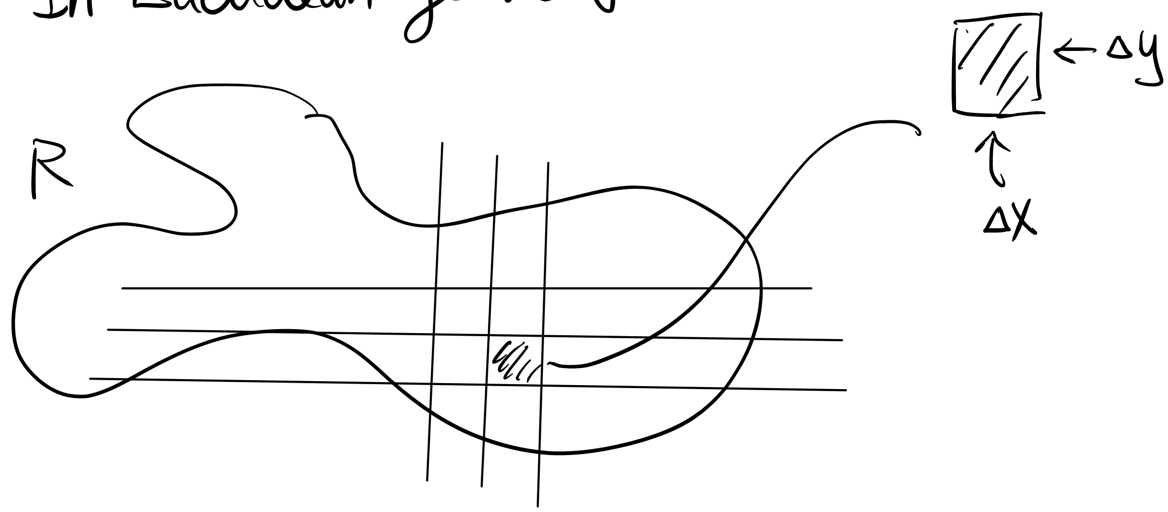
$$= \int_a^b \frac{|z'|}{y}\, dt$$

Remark: Hyperbolic straight lines in the upper half plane model are



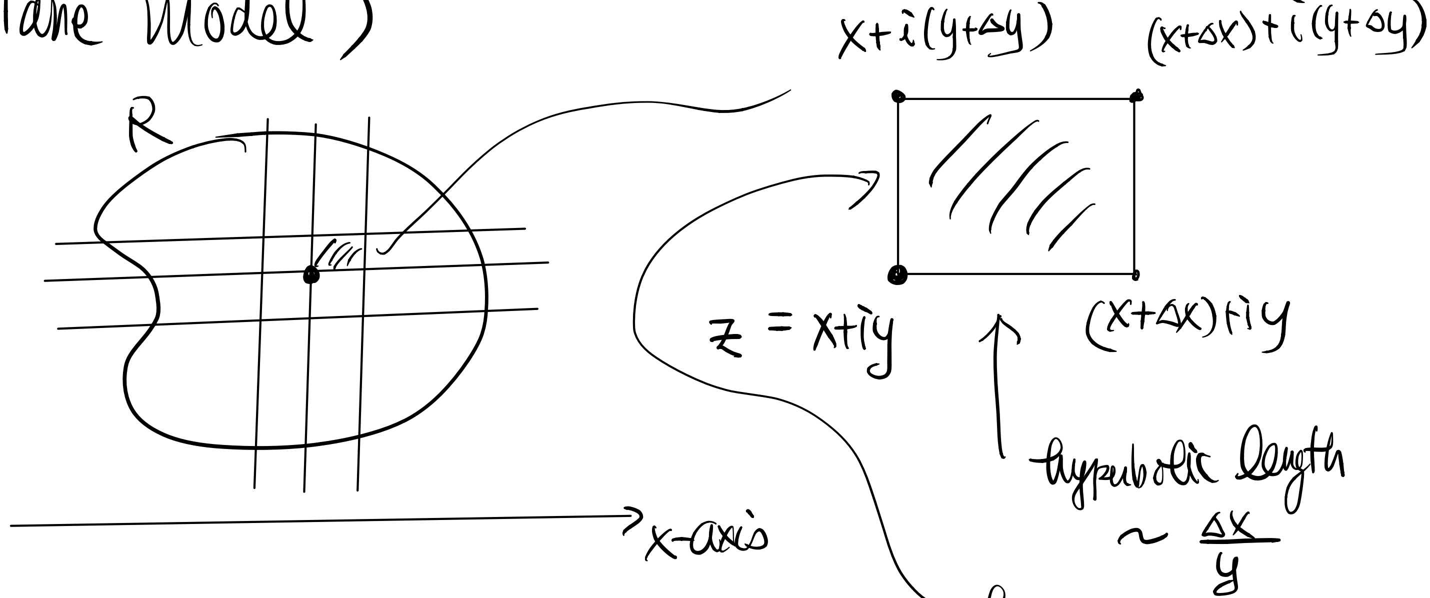
Ch10 Area (hyperbolic)

Recall: In Euclidean geometry



$$\text{Area}(R) \sim \sum \Delta x \Delta y \longrightarrow \iint_R dx\,dy$$

Similarly, in hyperbolic geometry (upper half plane model)



hyperbolic length $\sim \dfrac{\Delta x}{y}$

hyperbolic length $\sim \dfrac{\Delta y}{y}$

$$\left[\begin{array}{l} z(t) = x+it,\ y \le t \le y+\Delta y \\ \ell = \displaystyle\int_y^{y+\Delta y} \frac{|i|}{t}\,dt = \ln \frac{y+\Delta y}{y} \sim \frac{\Delta y}{y} \end{array}\right]$$

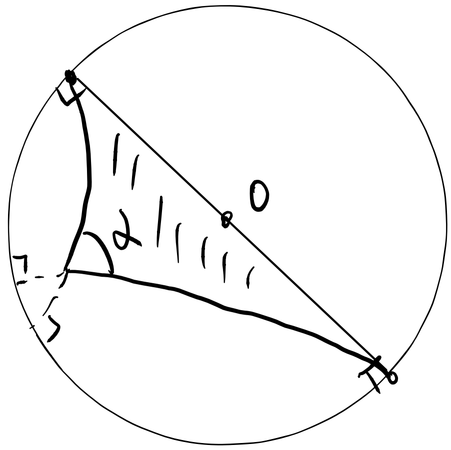
Hence, we make

Def: The (hyperbolic) area of a region R in the hyperbolic upper half plane model is given by

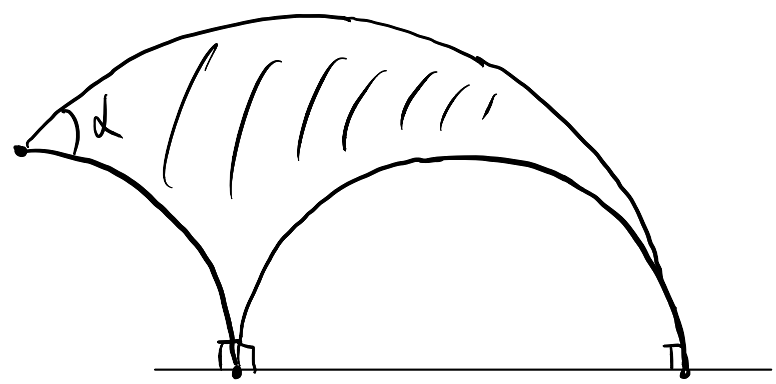
$$\boxed{A = \iint_R \frac{dx\,dy}{y^2}}$$

Areas of Triangles

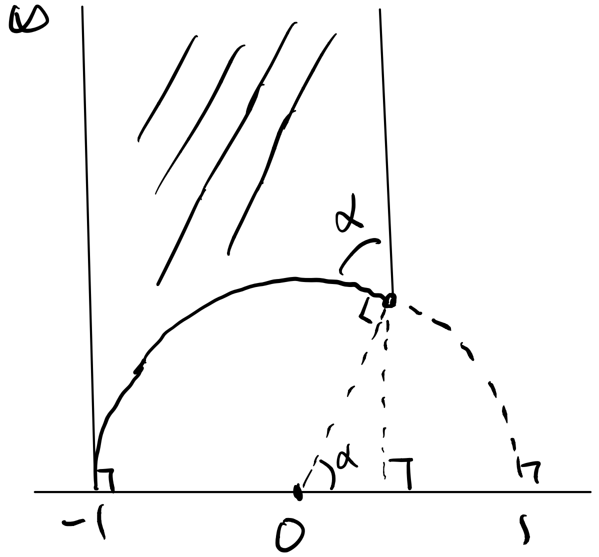
(1) Doubly asymptotic triangles
(i.e. triangles with 2 ideal vertexes)



disk model



upper half-plane model



We only need to consider the case that the ideal points at ∞ and -1, and the "finite" vertex somewhere along the unit circle

(Ex: Hints: horizontal translations and scaling are transformations of $(\overline{\mathbb{H}}, \mathbb{H})$)

Let α = interior angle of the triangle at the "finite" vertex. Then Euclidean geometry

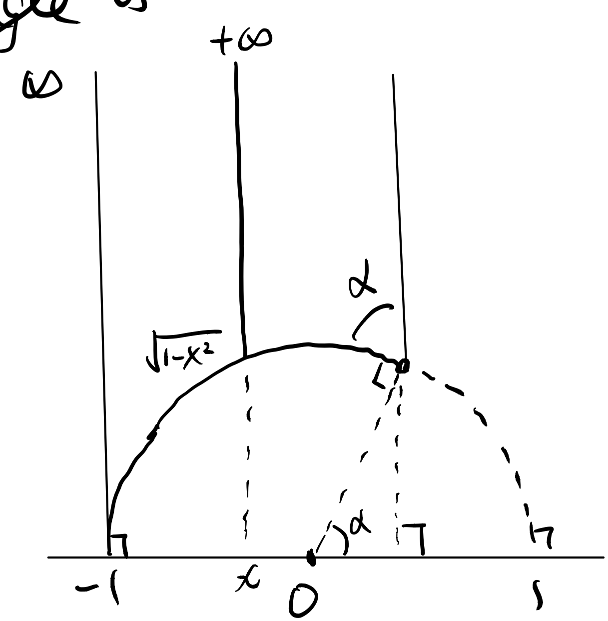
$\Rightarrow$ the "finite" vertex has coordinates

$$(\cos\alpha, \sin\alpha)$$

Hence the area of the triangle is

$$A = \iint \frac{dx\,dy}{y^2}$$

$$= \int_{-1}^{\cos\alpha} \left(\int_{\sqrt{1-x^2}}^{+\infty} \frac{dy}{y^2} \right) dx$$



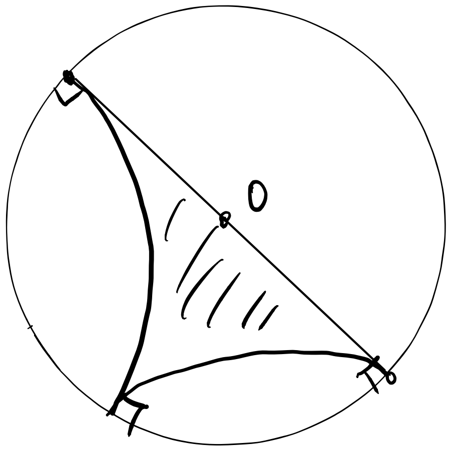
$$= \int_{-1}^{\cos\alpha} \frac{1}{\sqrt{1-x^2}}\,dx \quad \left(\text{let } x = \cos\theta,\ \theta \in [\alpha, \pi] \Rightarrow \sin\theta \geq 0 \ \& \ x' \leq 0\right)$$

$$= \pi - \alpha$$

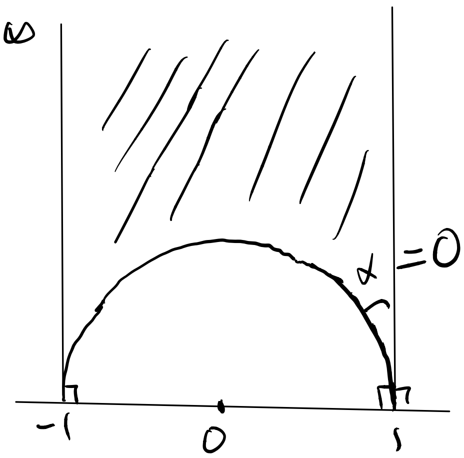
i.e. $\boxed{A = \pi - \alpha}$

(2) <u>Trebly asymptotic triangle</u> (<u>ideal triangle</u>)

(i.e. all 3 vertexes are ideal points.)



$(\alpha = 0)$



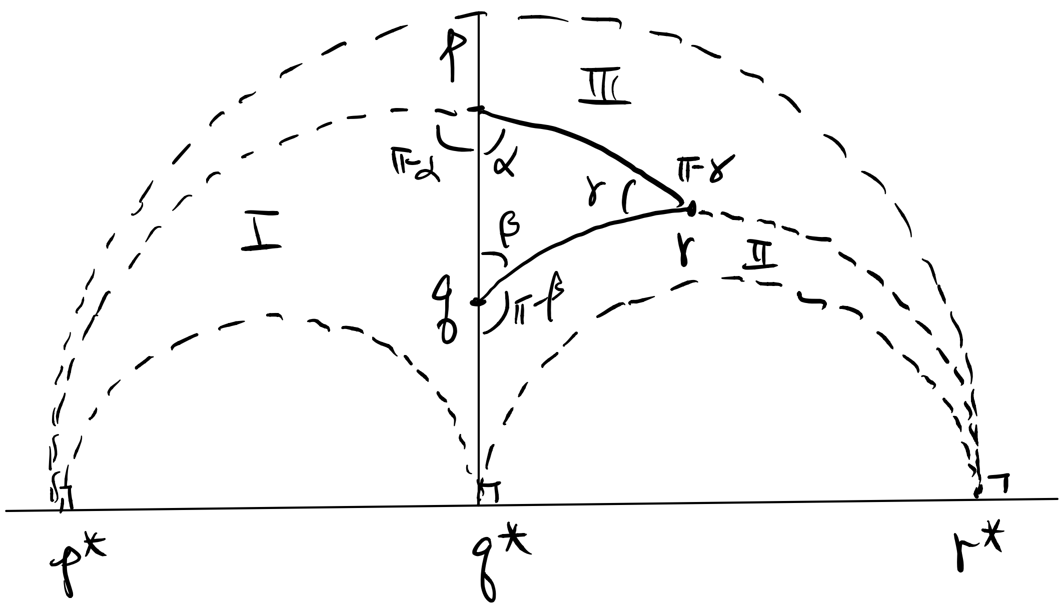
By (1), we have $\boxed{A = \pi}$

(for any trebly asymptotic triangle.)

(3) <u>General triangle</u>

We may put the triangle in a way such

that one of the edge is long the y-axis



$\mathrm{I} = \Delta p p^* q^*$, $\mathrm{II} = \Delta q q^* r^*$, $\mathrm{III} = \Delta r r^* p^*$

(as in the figure) are doubly asymptotic triangles.

Then by

$$\Delta p^* q^* r^* = \Delta pqr \cup \mathrm{I} \cup \mathrm{II} \cup \mathrm{III}$$

(interior disjoint union)

is a trebly asymptotic triangle

we have by (1) & (2)

$$\pi = A + [\pi - (\pi - \alpha)] + [\pi - (\pi - \beta)] + [\pi - (\pi - \gamma)]$$
$$= A + (\alpha + \beta + \gamma)$$

$$\Rightarrow \quad \boxed{A = \pi - (\alpha + \beta + \gamma)}$$

i.e. The <u>area</u> of a triangle equals to <u>π minus the sum of interior angles</u> which is called <u>angular defect</u>.

Thm: The <u>area</u> of a triangle (in hyperbolic geometry) equals its <u>angular defect</u>.

Thm: The sum of the interior angles of a triangles in hyperbolic geometry is less than π radians.