Case (A)

Let $p \in \mathbb{D}$ and $p^* \in \mathbb{C} \setminus \mathbb{D}$ be the fixed points of T (If $p = 0$, then $p^* = \infty$)

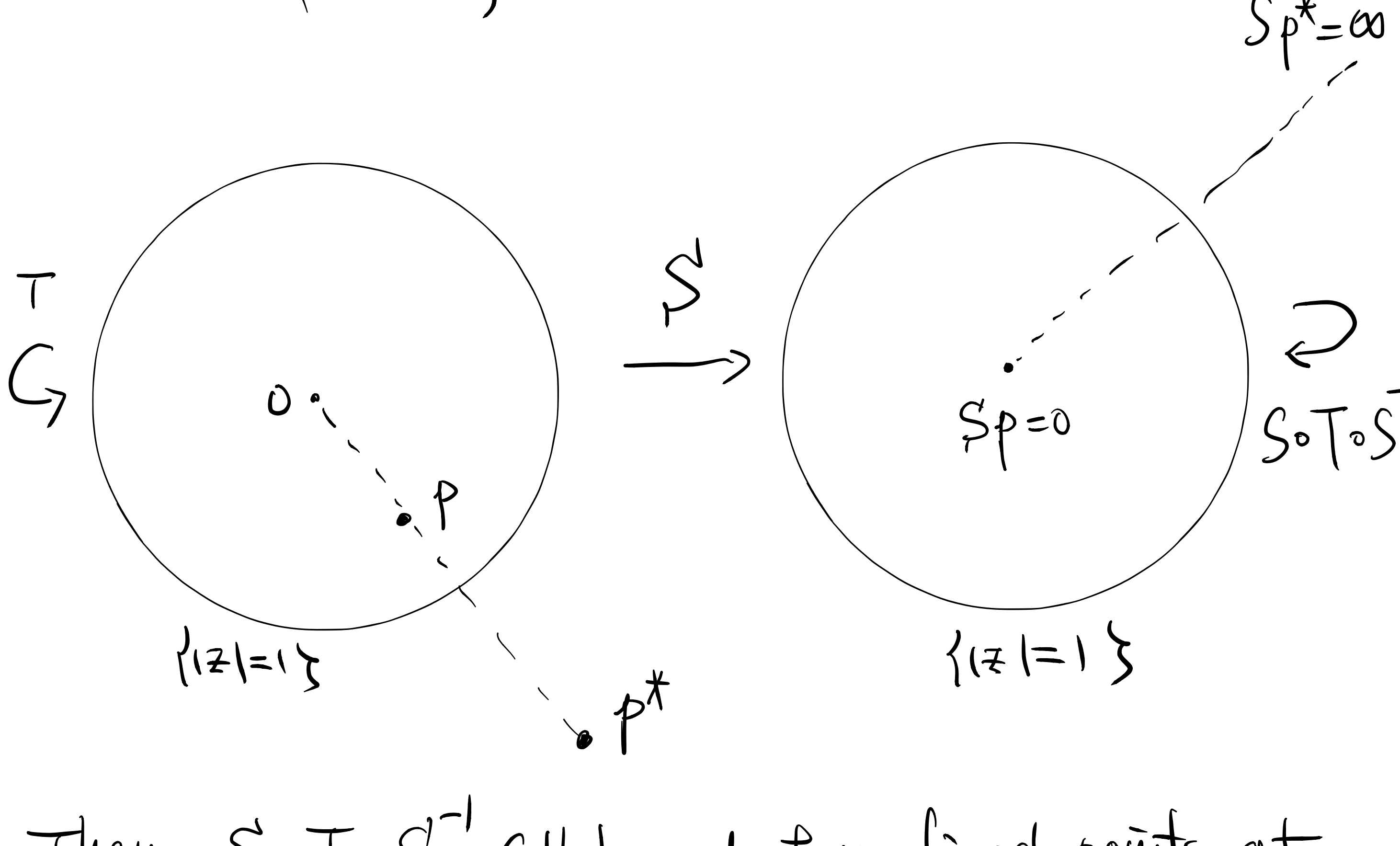
Let $S \in \mathbb{H}$ be a transformation in the hyperbolic group such that

$$Sp = 0$$

and hence

$$Sp^* = \infty.$$

Since $S \in \mathbb{H}$, $S(\{|z|=1\}) = \{|z|=1\}$.

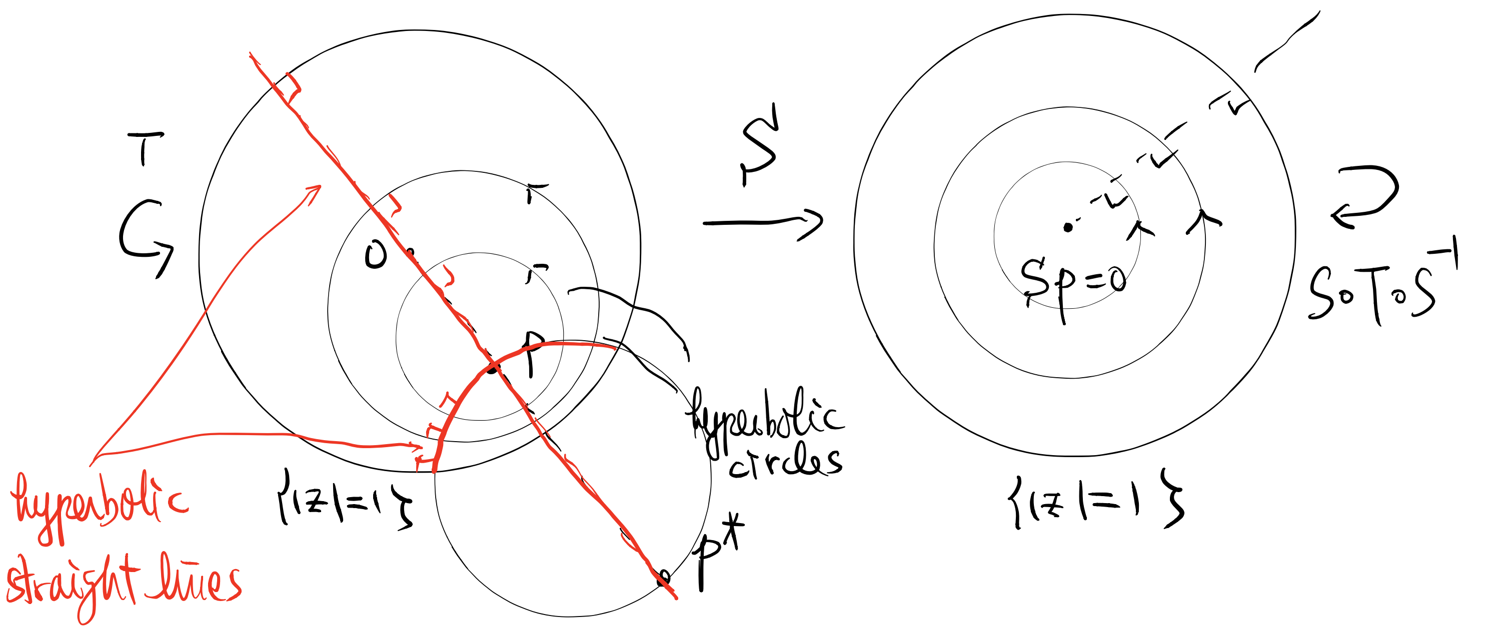


Then $S \circ T \circ S^{-1} \in \mathbb{H}$ and has fixed points at 0 and ∞, and fixing the unit circle.

Hence $S \circ T \circ S^{-1}$ is a rotation and T is a <u>elliptic</u> transformation. Moreover, we have the following conclusions:

(1) the unit circle is a Steiner circle of the 2^{nd} kind wrt p and p^*.

(2) Other Steiner circles of 2^{nd} kind (Apollonius) are hyperbolic circles.



(3) The transformation T "rotates" points around these circles of Apollonius, creating a circulating motion about the fixed point p.

(T is called a <u>hyperbolic rotation</u>)

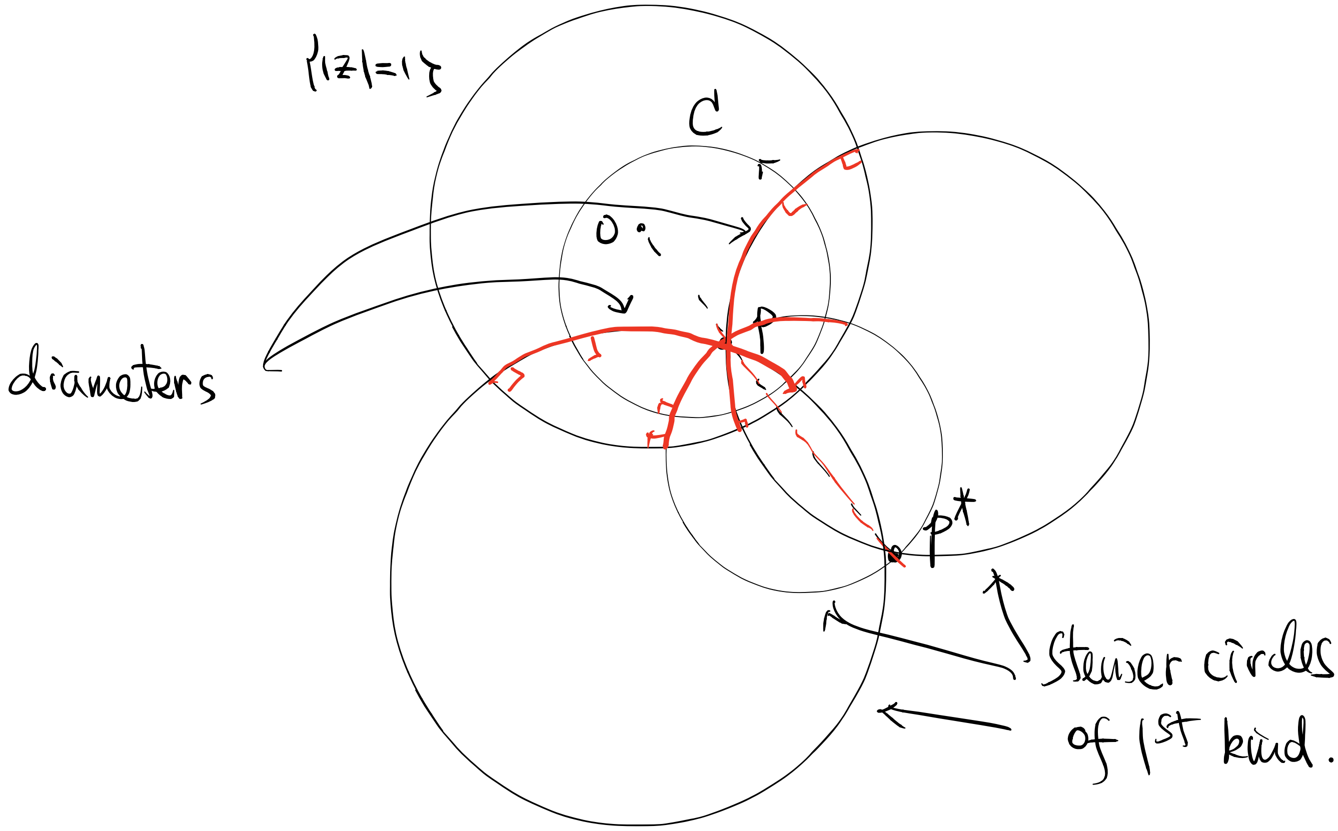
(4) The Steiner circles of 1^{st} kind wrt p and p^* are hyperbolic straight lines (passing thro. p)

<u>Remark</u>:

Suppose that C is a hyperbolic circle. Then the family of all circles <u>perpendicular</u> to C and <u>perpendicular</u> to the unit circle $\{|z|=1\}$ is a family of Steiner circles of the 1^{st} kind wrt <u>some points</u> p and p^* $(|p|<1, |p^*|>1)$.

(Ex!)

The point p is called the <u>center</u> of C, and the Steiner circles of the 1^{st} kind are called the <u>diameters</u> of C.



Case (B)

Let p and q be the 2 fixed points of T on $\{|z|=1\}$ $\Rightarrow$ T is hyperbolic

($\because$ elliptic with fixed p and q will move $\{|z|=1\}$)

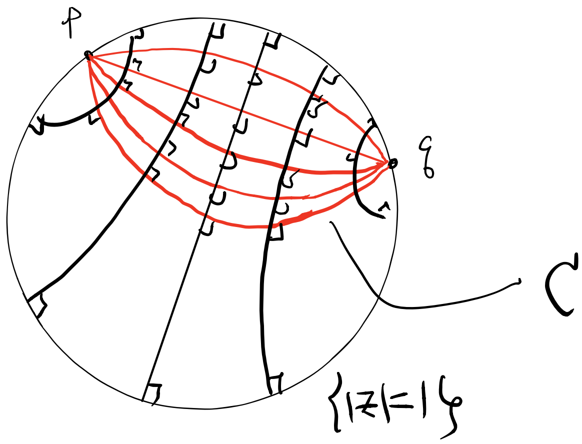
(1) Then the unit circle is a Steiner Circle of 1st kind wrt p and q.

(2) Other Steiner circles of 1st kind consist of

(a) a hyperbolic straight lines determined by p and q

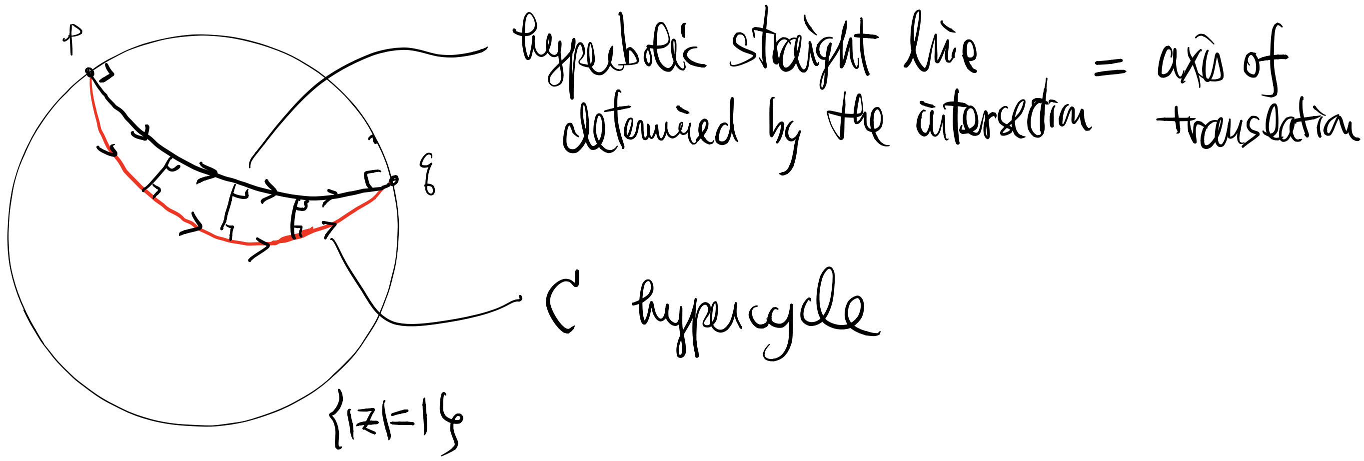
(b) hypercycles.

(3) Steiner circles of 2^{nd} kind are mutually hyperparallel hyperbolic straight lines



<u>Remark</u>: Conversely, if we have a hypercycle C. Then C intersects the unit circle in 2 points, say p & q $\Rightarrow$ C is a Steiner circle of the 1^{st} kind wrt p & q.

(unit circle & the hyperbolic straight line determined by p & q are also Steiner circles of the 1^{st} kind)



The "hyperbolic distance" (see next section) of the perpendicular from C to the hyperbolic straight line is <u>constant</u> (i.e. independent of the starting point on C) since these perpendiculars are <u>congruent</u>. (Ex!)

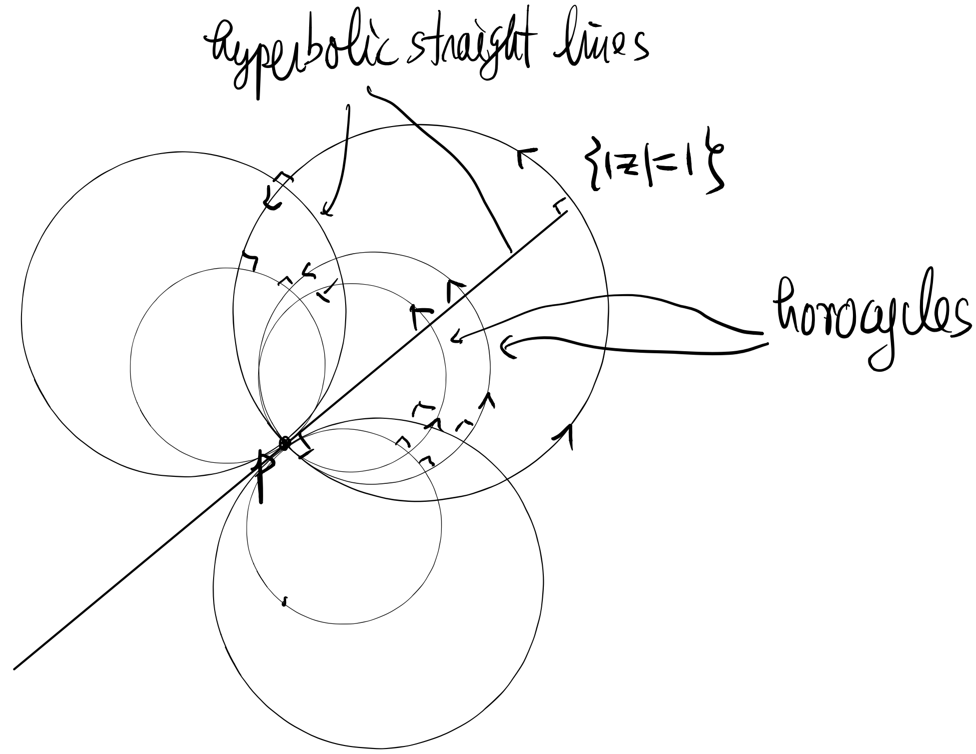
- Therefore, <u>hypercycles</u> are called <u>equidistant curves</u>.

  (Ex: Compare with Euclidean geometry!)

<u>Note</u> that T moves points along the equidistant curves from to p to q (or q to p), and hence called a <u>hyperbolic translation</u>; the unique hyperbolic straight line determined by p & q is called the <u>axis of translation</u>.

Case (C)

T must be a parabolic transformation



(1) the hyperbolic straight lines passing thro. p are degenerated Steiner circles perpendicular to {|z|=1}

(2) The family of degenerate Steiner circles perpendicular to the hyperbolic straight lines from p are horocycles.

(3) The hyperbolic straight lines thro. p are the diameters of the horocycles.

(4) The transformation T is called a parallel displacement.

Summary:

(A) hyperbolic circle is a curve traced out by a point subjected to elliptic transformation

$\Leftrightarrow$ Steiner circle of 2nd kind wrt the fixed points of an elliptic transformation (fixing $\{|z|=1\}$)

(B) hypercycle is a curve traced out by a point subjected to hyperbolic transformation

$\Leftrightarrow$ Steiner circle of 1st kind wrt the fixed points of a hyperbolic transformation (fixing $\{|z|=1\}$)

(C) horocycle is a curve traced out by a point subjected to parabolic transformation.

$\longleftrightarrow$ <u>degenerate</u> Steiner circle of a parabolic transformation (fixing $\{|z|=1\}$) perpendicular to hyperbolic straight lines thro. the fixed (ideal) point.

Ch9 Hyperbolic length

Def: A (parametric) curve

$$\gamma = z(t) = x(t) + i\,y(t), \quad t \in [a,b]$$

is called smooth if $x(t)$ and $y(t)$ are differentiable.

- $z(a)$ and $z(b)$ are called end points of the curve γ.

Def: In the hyperbolic plane, the length of a smooth curve γ with parametrization $z(t) = x(t) + iy(t)$ $a \le t \le b$, is given by

$$\boxed{\ell(\gamma) = 2\int_a^b \frac{|z'(t)|}{1-|z(t)|^2}\,dt}$$

where $z'(t) = x'(t) + i\,y'(t)$

Def: Let z_1 and z_2 be two points in the hyperbolic plane. The distance from z_1 to z_2 is defined by

$$d(z_1, z_2) = \ell\left(\begin{array}{c}\text{hyperbolic straight line segment} \\ \text{between } z_1 \text{ \& } z_2\end{array}\right)$$

Remarks: (i) $\ell(\gamma) \geq 0, \ \forall \gamma$.

(ii) $|z'(t)|dt = \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$

is just the usual Euclidean integrand for arc-length.

Thm: Let T be a transformation of hyperbolic group and γ be a smooth curve. Then

$$\ell(T(\gamma)) = \ell(\gamma).$$

i.e. "length" is invariant

$\Rightarrow$ "distance" is invariant.

Pf: Let $w = Tz = e^{i\theta} \dfrac{z - z_0}{1 - \bar{z}_0 z}$ $\quad (|z_0|<1, \theta \in \mathbb{R})$

and $\gamma : z(t)$

Then $T(\gamma)$: $w(t) = Tz(t)$

$$= e^{i\theta} \frac{z(t) - z_0}{1 - \bar{z}_0 z(t)}.$$

And $w'(t) = e^{i\theta} \dfrac{1 - |z_0|^2}{[1 - \bar{z}_0 z(t)]^2} z'(t)$ $\quad$ (Ex!)

[Note: $T'(z) = e^{i\theta} \dfrac{1 - |z_0|^2}{(1 - \bar{z}_0 z)^2}$]

$$\Rightarrow \quad \frac{|w'(t)|}{1 - |w(t)|^2} = \frac{1}{1 - \left| \dfrac{z(t) - z_0}{1 - \bar{z}_0 z(t)} \right|^2} \cdot \frac{1 - |z_0|^2}{|1 - \bar{z}_0 z(t)|^2} |z'(t)|$$

$$= \frac{1 - |z_0|^2}{|1 - \bar{z}_0 z(t)|^2 - |z(t) - z_0|^2} |z'(t)|$$

$$= \frac{1 - |z_0|^2}{\left[\begin{array}{l} 1 - \bar{z}_0 z(t) - z_0 \overline{z(t)} + |z_0|^2 |z(t)|^2 \\ - |z(t)|^2 + \bar{z}_0 z(t) + z_0 \overline{z(t)} - |z_0|^2 \end{array}\right]} |z'(t)|$$

$$= \frac{1-|z_0|^2}{1-|z_0|^2-|z(t)|^2+|z_0|^2|z(t)|^2}\,|z'(t)|$$

$$= \frac{1-|z_0|^2}{(1-|z_0|^2)(1-|z(t)|^2)}\,|z'(t)|$$

$$= \frac{|z'(t)|}{1-|z(t)|^2}$$

Hence $\ell(T\gamma) = 2\int_a^b \frac{|w'(t)|}{1-|w(t)|^2}\,dt$

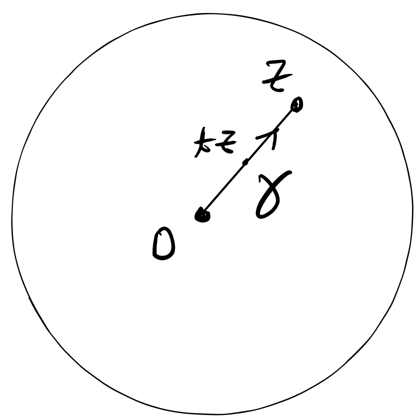
$$= 2\int_a^b \frac{|z'(t)|}{1-|z(t)|^2}\,dt = \ell(\gamma).$$

#

Distance famula

Case 1 $\boxed{d(0,z) = \ln\frac{1+|z|}{1-|z|}} \boxed{= \log\frac{1+|z|}{1-|z|}}$

Pf: The Euclidean straight line segment is the hyperbolic straight line segment in this case. Hence $d(0,z) = \ell(\gamma)$



where $\gamma: z(t) = tz, \quad 0 \le t \le 1.$

$$\Rightarrow \quad d(0,z) = 2\int_0^1 \frac{|z'(t)|}{1-|z(t)|^2}\,dt = 2\int_0^1 \frac{|z|}{1-|tz|^2}\,dt$$

$$= 2\int_0^{|z|} \frac{dr}{1-r^2}, \quad \text{letting } r = t|z|$$

$$= \ln \frac{1+|z|}{1-|z|}. \quad \#$$

Case 2 (general case)

$$\boxed{d(z_1,z_2) = \ln\left(\frac{1+\left|\frac{z_2-z_1}{1-\bar{z}_1 z_2}\right|}{1-\left|\frac{z_2-z_1}{1-\bar{z}_1 z_2}\right|}\right)}$$

Pf: Let $Tz = e^{i\theta}\dfrac{z-z_1}{1-\bar{z}_1 z}$ (any θ)

Then $Tz_1 = 0$

By invariance of ℓ, we have

$$d(z_1,z_2) = d(Tz_1, Tz_2)$$

$$= d(0, Tz_2)$$

$$= \ln \frac{1+|Tz_2|}{1-|Tz_2|} \quad (\text{by case } 1)$$

$$= \ln \frac{1+\left|\frac{z_2-z_1}{1-\bar{z}_1 z_2}\right|}{1-\left|\frac{z_2-z_1}{1-\bar{z}_1 z_2}\right|}$$

#

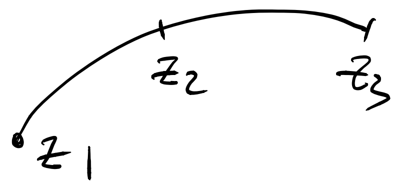
Fundamental Properties of Distance

Thm: Let z_1, z_2, z_3 be points in the hyperbolic plane.

Then (1) $d(z_1, z_2) \geq 0$

(2) $d(z_1, z_2) = d(z_2, z_1)$

(3) If z_1, z_2 and z_3 are colinear (in the order)



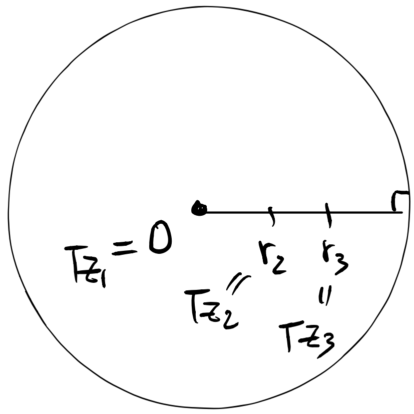
then

$$d(z_1, z_3) = d(z_1, z_2) + d(z_2, z_3).$$

(colinear : z_1, z_2, z_3 on the same hyperbolic straight line)

Pf: (1) & (2) are clear.

Pf of (3): Let T be a transformation of hyperbolic geometry taking z_1 to 0 and the hyperbolic straight line passing thro, z_1, z_2 & z_3 to the (positive) x-axis. Then $Tz_1 = 0$, $Tz_2 = r_2$ and $Tz_3 = r_3 > r_2$ (with $r_2, r_3 \in \mathbb{R}$)



Then

$$d(z_1, z_3) = d(0, r_3)$$
$$= 2\int_0^{r_3} \frac{dr}{1-r^2}$$
$$= 2\int_0^{r_2} \frac{dr}{1-r^2} + 2\int_{r_2}^{r_3} \frac{dr}{1-r^2}$$
$$= d(0, r_2) + 2\int_{r_2}^{r_3} \frac{dr}{1-r^2}$$

Note $d(z_2, z_3) = d(r_2, r_3) = \ln \dfrac{1 + \left|\frac{r_3 - r_2}{1 - \overline{r_2} r_3}\right|}{1 - \left|\frac{r_3 - r_2}{1 - \overline{r_2} r_3}\right|}$

$$= \ln \frac{1+\left|\frac{r_3-r_2}{1-r_2r_3}\right|}{1-\left|\frac{r_3-r_2}{1-r_2r_3}\right|} = \ln \frac{1-r_2r_3+r_3-r_2}{1-r_2r_3-r_3+r_2}$$

(since $0<r_2<r_3<1$)

$$= \ln \frac{(1+r_3)(1-r_2)}{(1-r_3)(1+r_2)} = \ln \frac{1+r_3}{1-r_3} - \ln \frac{1+r_2}{1-r_2}$$

$$= 2\int_{r_2}^{r_3} \frac{dr}{1-r^2} .$$

$\therefore\ d(z_1,z_3) = d(z_1,z_2) + d(z_2,z_3)$. ※

Thm Let z_1 & z_2 be points in the hyperbolic plane. Then the shortest curve (in hyperbolic length) connecting z_1 & z_2 is the hyperbolic straight line segment joining z_1 & z_2.