Thm

(i) In hyperbolic geometry, all hyperbolic straight lines are congruent.

(ii) Two points in the hyperbolic geometry determine a unique hyperbolic straight line.

Recall: " $T(z^{*}) = (Tz)^{*}$ " in our case of

$\underset{\text{wrt } \{|z|=1\}}{z^{*}}$ $\quad$ $\underset{\text{wrt } \{|z|=1\}}{(Tz)^{*}}$

(T = transformation in hyperbolic group)

of unit circle can be written as

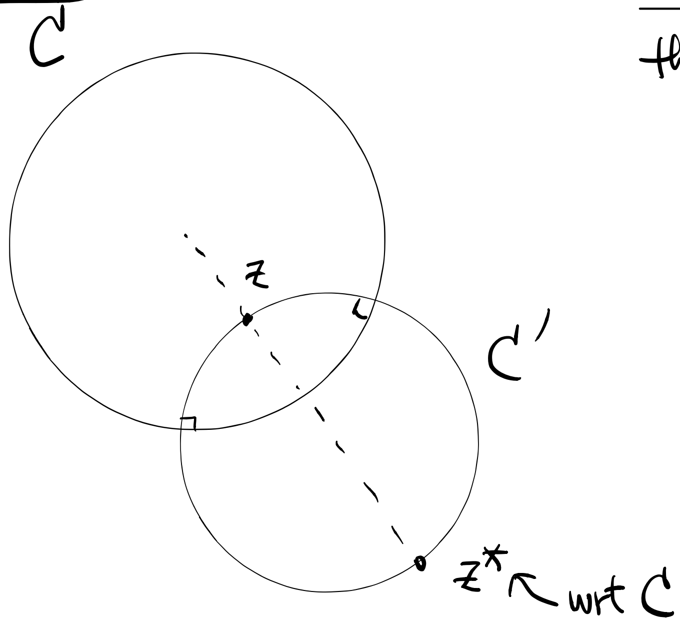
Lemma 1: Each transformation of hyperbolic geometry maps each pair of points symmetric wrt the unit circle to another pair of points symmetric wrt the unit circle.

We also need the following :

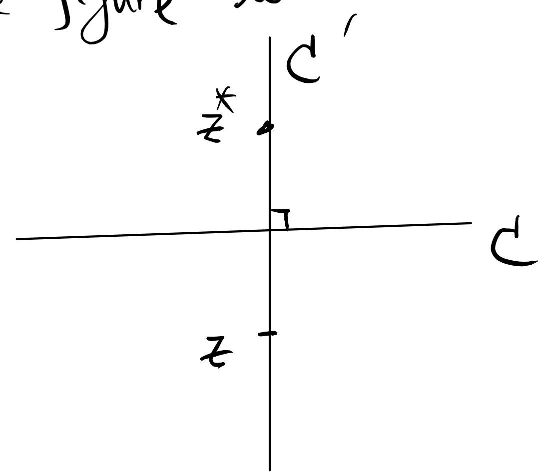
Lemma 2: Let C be a cline. Let z & z^* be distinct symmetric points wrt C. Then any cline C' that is athogonal to C and passing throught z must also passing throught z^*.

Conversely, any cline that passes throught z & z^* is orthogonal to C.

Pf of Lemma 2



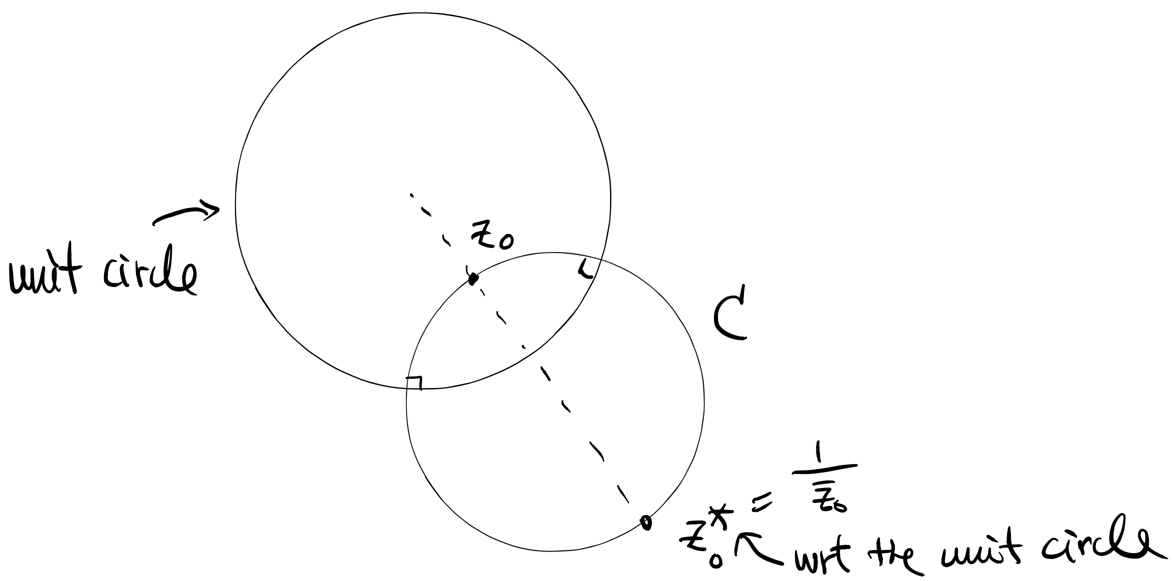
Idea of proof: transfams the figure to



Proof of the Thm:

(i) Let C be a cline that is athogonal to the unit circle, i.e. C is a hyperbolic straight line

And let z_0 be a point on C.



By Lemma 2, z_0^* also lies on C (outsides $\mathbb{D}$), where $*$ = sym. wrt the unit circle.

Let $Tz = e^{i\theta} \dfrac{z - z_0}{1 - \bar{z}_0 z}$ (θ chosen later)

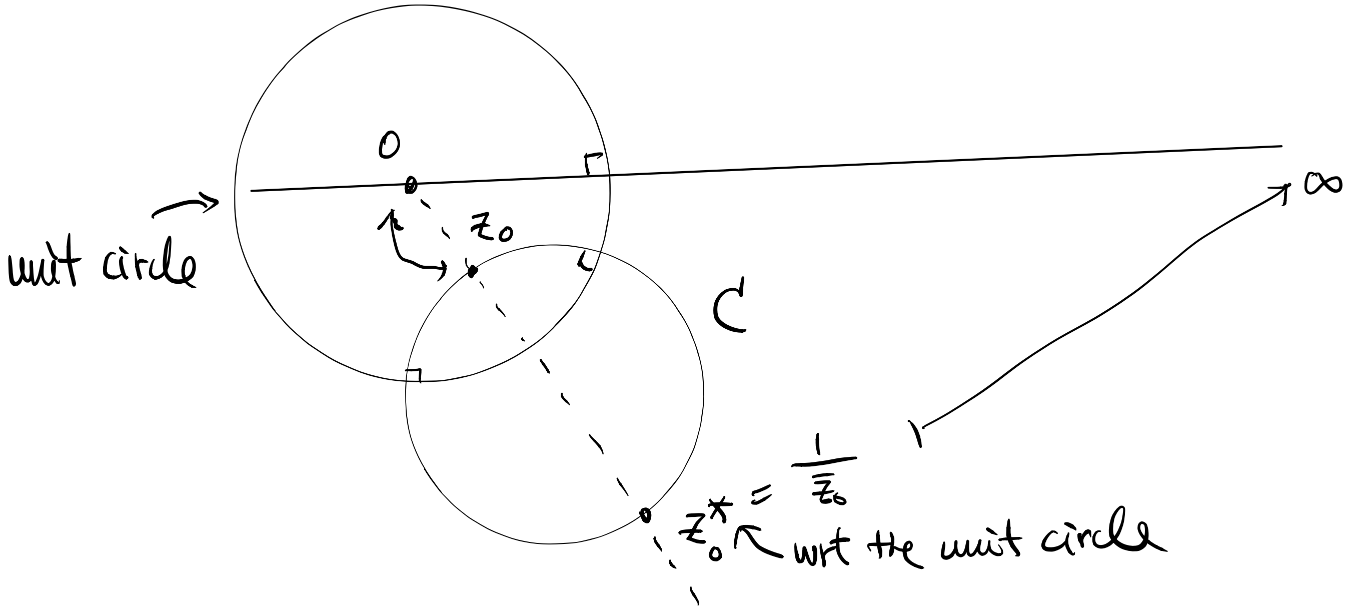
Then T is a transformation of hyperbolic geometry (i.e. $T \in \mathbb{H}$ the hyperbolic group) and

$$Tz_0 = 0, \quad T(z_0^*) = T\left(\frac{1}{\bar{z}_0}\right) = \infty$$

$\therefore$ $T(C)$ is a cline passing throught 0 & ∞, and orthogonal to $\{|z| = 1\}$.

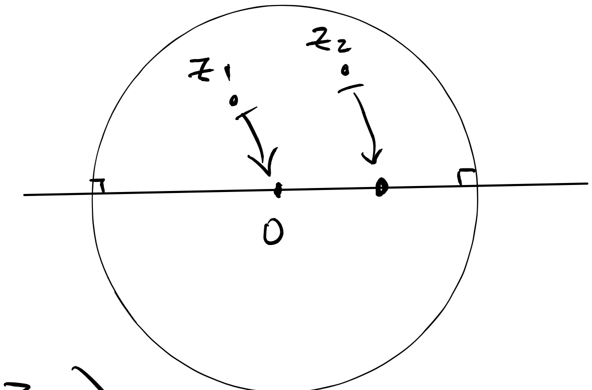
$\Rightarrow$ $T(C)$ must be a diameter of the unit circle

Finally, we can choose θ so that $T(C) = x$-axis. (Ex!)



This proves that any hyperbolic straight line C is congruent to the x-axis. And hence all hyperbolic straight lines are congruent. This proves part (i).

For part (ii), let z_1, z_2 be any 2 distinct points in $\mathbb{D}$. Then



$$Tz = e^{i\theta} \frac{z - z_1}{1 - \bar{z}_1 z}$$

takes z_1 to 0.

Choose $\theta = -\arg\left(\frac{z_2 - z_1}{1 - \bar{z}_1 z_2}\right)$

Then $Tz_2 = e^{i\theta} \frac{z_2 - z_1}{1 - \bar{z}_1 z_2}$

$$= e^{i\theta}\left|\frac{z_2-z_1}{1-\bar{z}_1 z_2}\right| e^{i\arg\left(\frac{z_2-z_1}{1-\bar{z}_1 z_2}\right)}$$

$$= \left|\frac{z_2-z_1}{1-\bar{z}_1 z_2}\right| > 0 \quad \left(\begin{array}{l}\text{positive real number,} \\ \text{and in fact } <1.\end{array}\right)$$

Note that x-axis is the unique hyperbolic straight line passing through 0 and Tz_2. This proves that T^{-1}(x-axis) is the unique hyperbolic straight line passing throught z_1 & z_2. ##

Euclid's Postulates

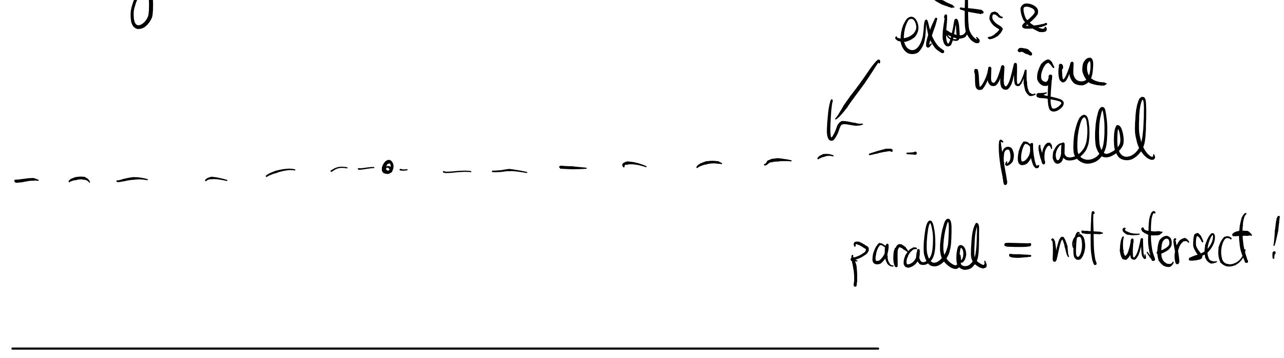
Postulate 1 : Two points determine a straight line.

Postulate 2 : A line can be produced indefinitely in either direction.

Postulate 3 : A circle can be described with any center and radius.

Postulate 4 : All right angles are congruent.

Postulate 5: Through a point not on a line, there is a unique line parallel to the given line.

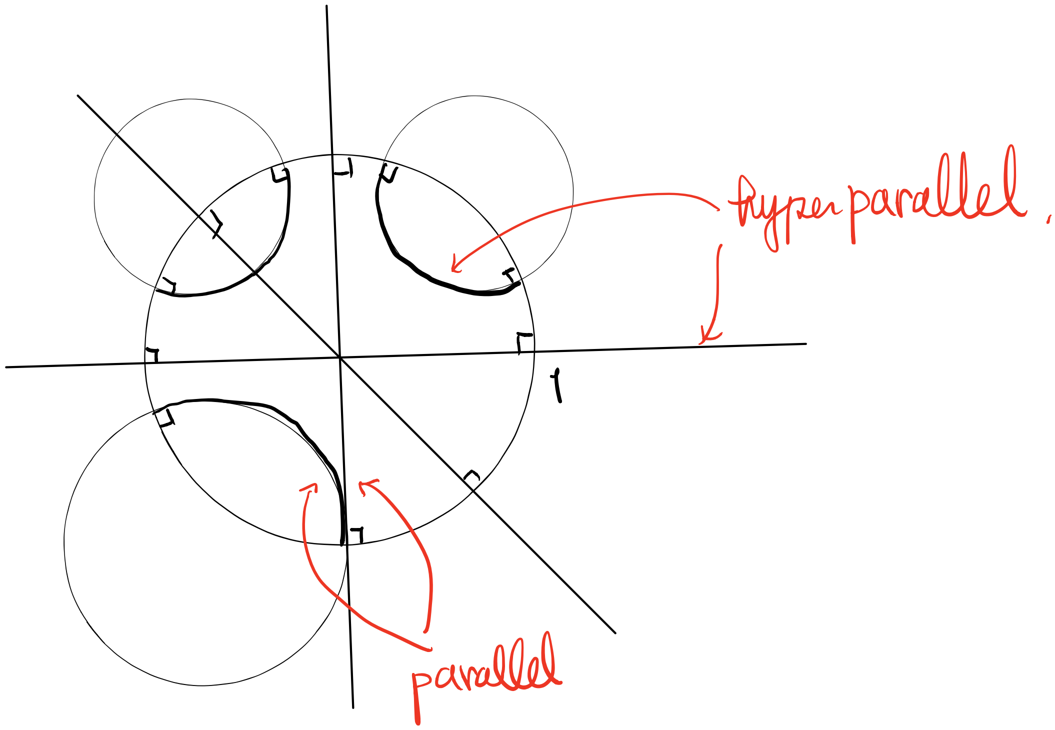


Parallelism in hyperbolic geometry

Def: (i) The points on the unit circle are called ideal points.

(ii) Two hyperbolic lines are called parallel if they do not intersect inside $\mathbb{D}$ but do share one ideal point.

(iii) Two hyperbolic lines are called hyperparallel if they do not intersect inside $\mathbb{D}$ and do not have an ideal point in common.

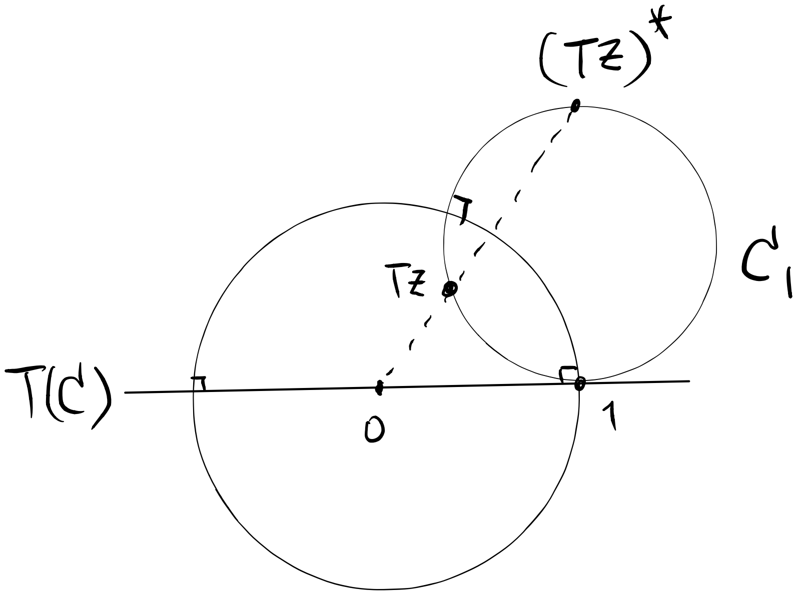


$\boxed{\text{Postulate 5 is false in hyperbolic geometry.}}$

In fact, $\forall$ point p not on a hyperbolic line C, there exists 2 hyperbolic lines parallel to C and passing thro. p.

Pf: For any hyperbolic line C, $\exists$ transformation $T \in \mathbb{H}$ such that $T(C) = x\text{-axis}$.

If $z \in \mathbb{D}$ is a point not on C, then Tz is a point not on $T(C) = x\text{-axis}$



Let C_1 be the Euclidean circle passing thro. the points 1, Tz, $(Tz)^*$

Lemma2 $\Rightarrow$ C_1 is orthogonal to the unit circle.

Since C'_1 & $T(C') = x$-axis tangent at 1, they have no other intersection.

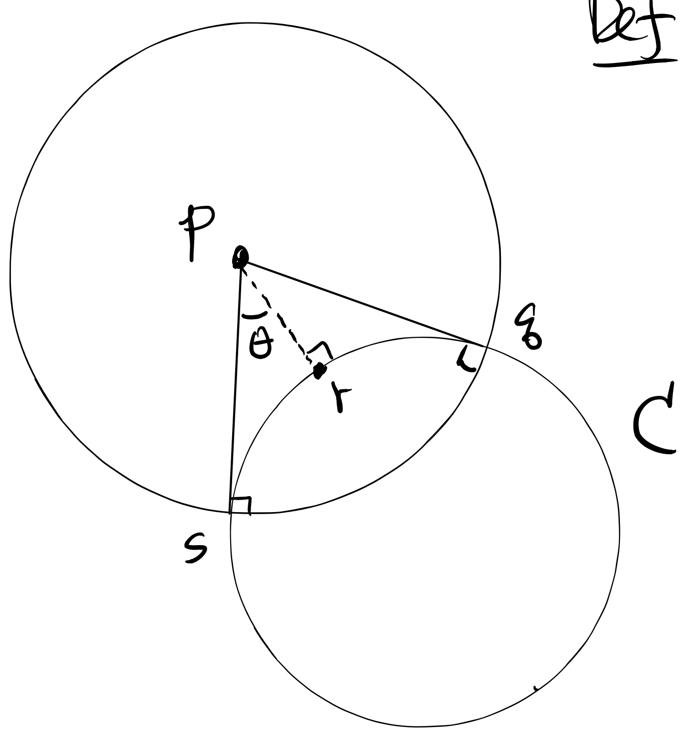
$\Rightarrow$ C'_1 is parallel to $T(C')$ and passing thro. Tz.

$\Rightarrow$ $T^{-1}(C'_1)$ is parallel to C' and passing thro. z.

Similarly, we can find C'_{-1} passing thro -1, Tz, $(Tz)^*$.
and $T^{-1}(C'_{-1})$ is parallel to C' and passing thro. z.

Since $C'_1 \neq C'_{-1}$, Postulate 5 is false. ※

Angle of parallelism



Def. Let $C = \overline{srq}$ be a hyperbolic straight line and p be a point not on C such that the hyperbolic straight line passing thro p & r is perpendicular to C.

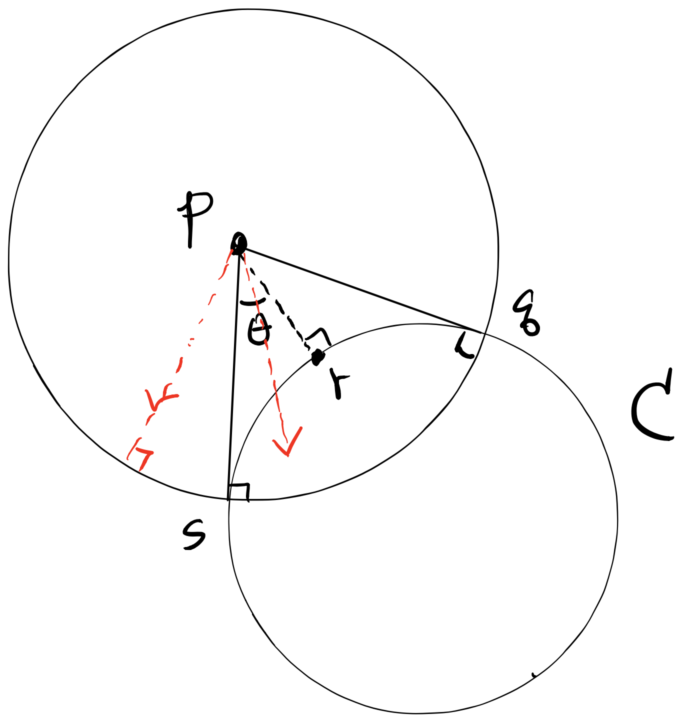
Then the angle θ between one of the parallels ($\overline{ps}$ or $\overline{pq}$) and the perpendicular $\overline{pr}$ is called the <u>angle of parallelism</u>.

<u>Remark</u> (Ex!): A <u>ray</u> passing thro. p makes

(i) an angle with $\overline{pr} < \theta$, then it <u>intersects</u> $\overline{srq}$.

(ii) an angle with $\overline{pr} = \theta$, then it is <u>parallel</u> to $\overline{srq}$.

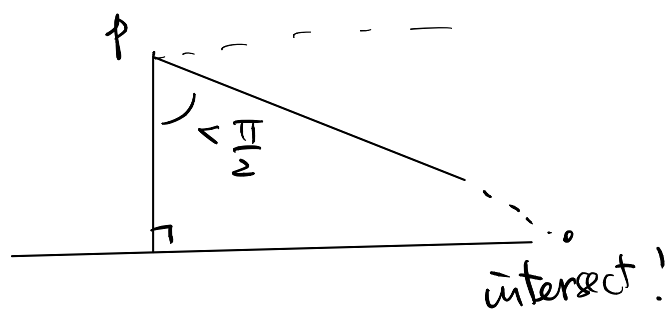
(iii) an angle with $\overline{pr} > \theta$, then it is <u>hyperparallel</u> to $\overline{srq}$.



<u>Remark</u>: The angle of parallelism is always <u>acute</u>.

(Easy Ex: transform p to 0)

[<u>Compare</u>: Euclidean geometry



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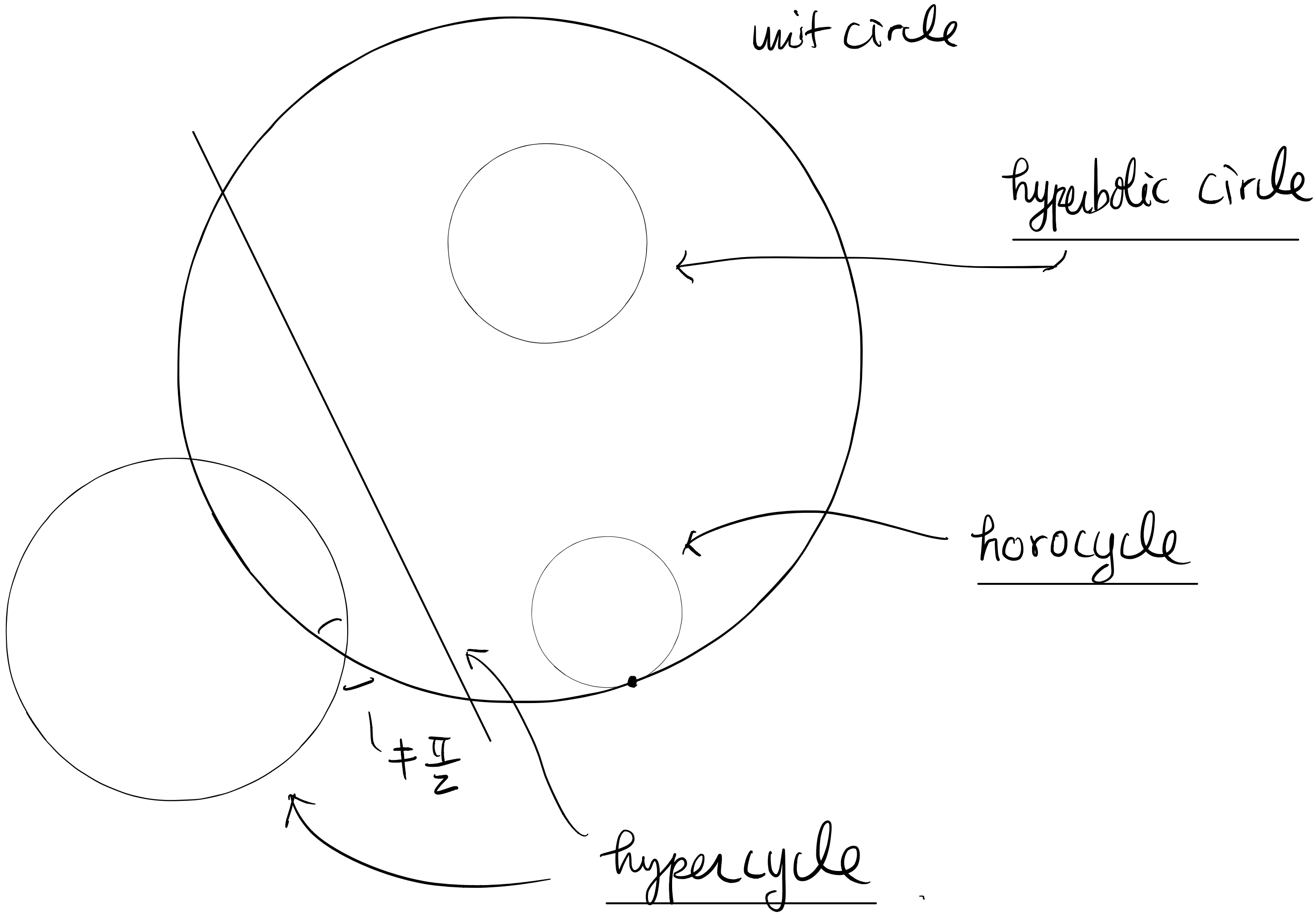
Ch8 Cycles



Def: Let C be a portion of a Euclidean circle or straight line inside the unit disk. Suppose that C is not perpendicular to the unit circle. Then C is called a cycle.

- If C is entirely contained in $\mathbb{D}$, then C is a hyperbolic circle.
- If C is tangent to the unit circle, then

C is a <u>horocycle</u>.

- If C <u>intersects</u> the unit circle (at an angle $\neq \frac{\pi}{2}$) C is a <u>hypercycle</u>.

By Lemma 1, if T is a transformation of the hyperbolic group, then T maps $\mathbb{D}$ onto itself and hence $T(\partial\mathbb{D}) = \partial\mathbb{D}$. If T has a <u>fixed point inside $\mathbb{D}$</u>, then the symmetric point (wrt $\partial\mathbb{D}$) <u>outside</u> $\mathbb{D}$ is also a fixed point of T: $T(z^*) = (Tz)^* = (z)^*$.

So we can analyze $T \in \mathbb{H}$ $(T \neq Id)$ by the following situations (using <u>cycles</u>)

(A) 1 fixed point inside $\mathbb{D}$ & 1 fixed point outside $\mathbb{D}$

(B) 2 fixed points on $\{|z|=1\} = \partial\mathbb{D}$.

(C) 1 fixed point only (must be on $\{|z|=1\} = \partial\mathbb{D}$)