THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT5120 Topics in Geometry 2018/19
Homework 1 Questions, solutions and remarks

Questions:

1. Show that $(\infty, z_1, z_2, z_3) = \frac{z_1 - z_3}{z_1 - z_2}$.
2. Find a Möbius transformation sending $0, i, -1$ to $-i, 1, 0$ respectively.
3. Find all Möbius transformations with fixed points i and $-i$.
4. Using Fundamental Theorem of Möbius geometry, show that all clines are congruent in Möbius geometry.
5. Let z be a point inside the circle $C : |z - a| = R$. Suppose that p, q be the two distinct points on the C such that the line segment $\overline{pq}$ passing through z and is perpendicular to $\overline{az}$. Show that the tangents to C at p and q meet at z^* (symmetric point of z wrt C).

Solutions:

1. $(\infty, z_1, z_2, z_3) = \frac{\infty - z_2}{\infty - z_3}\frac{z_1 - z_3}{z_1 - z_2} = \frac{z_1 - z_3}{z_1 - z_2}$.
2. Let $Tz = \frac{az+b}{cz+d}$ be a Möbius transformation sending $(0, i, -1)$ to $(-i, 1, 0)$. So we have $T(0) = \frac{b}{d} = -i$, $T(i) = \frac{ai+b}{ci+d} = 1$, $T(-1) = \frac{-a+b}{-c+d} = 0$.
Solving $b = -id$, $ai + b = ci + d$, $-a + b = 0$, we get $b = a$, $d = ia$, $c = -ia$.
So we get $T(z) = \frac{z+1}{-iz+i}$, which indeed sends $(0, i, -1)$ to $(-i, 1, 0)$.
3. We will provide two solutions. The first one is the quick and obvious way: Plug all the numbers in and solve for a, b, c, d. The second one is slower, and produces a messier answer, but gives some geometric insights.

   Solution 1:
   Let $T(z) = \frac{az+b}{cz+d}$ $(ad - bc \neq 0)$ be a Möbius transformation.
   T has fixed points $i, -i$
   $\Leftrightarrow i = \frac{ai+b}{ci+d}$ and $-i = \frac{-ai+b}{-ci+d}$
   $\Leftrightarrow -c + di = ai + b$ and $-c - di = -ai + b$
   $\Leftrightarrow a = d$ and $b = -c$ (by Gaussian elimination or whatever algebraic methods)
   So the general form of Möbius transformations with fixed points i and $-i$ is $T(z) = \frac{az+b}{-bz+a}$ $(a^2 + b^2 \neq 0)$

   WARNING: This warning is about logic. It is important that two conditions connected by a $\Leftrightarrow$ are actually equivalent. If you simply add $-c+di = ai+b$ and $-c-di = -ai+b$ together to get $2ai = 2di$ and then proceed to the next step, instead of finding an equivalent condition like $\begin{cases} 2ai = 2di \\ -2c = 2b \end{cases}$, then you may end up with some fake answers, i.e. some Möbius transformations in your final answer may not actually have both $i, -i$ as fixed points. If you don't understand why does that happen, try to think why do we get a fake solution “x=-2" if we try to solve the equation “x+1=1" in the following way:

$$x + 1 = 1$$
$$(x + 1)^2 = 1^2$$
$$x^2 + 2x + 1 = 1$$
$$x(x + 2) = 0$$
$$x = 0 \text{ or } x = -2$$

Solution 2:
We know that Möbius transformations that fix 0 and ∞ are in the form $T(z) = \lambda z$ for some non-zero complex number λ. It would be nice if we can pretend that the two points we're trying to fix are $0, \infty$. It can be done by moving $i, -i$ to $0, \infty$.
Let $S(z) = \frac{z-i}{z+i}$ (a Möbius transformation that moves $i, -i$ to $0, \infty$)
Consider the commutative diagram

$$\begin{array}{ccc} \widehat{\mathbb{C}} & \xrightarrow{T} & \widehat{\mathbb{C}} \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle S} \\ \widehat{\mathbb{C}} & \xrightarrow{R} & \widehat{\mathbb{C}} \end{array}$$

T fixes $i, -i$ if and only if R fixes $0, \infty$. So a Möbius transformaiton T fixes $i, -i$ if and only if it is in the form $T = S^{-1}RS$, where $R(z) = \lambda z$ for some non-zero complex number λ. Hence the general form of a Möbius transformation that fixes $i, -i$ is

$$\begin{aligned} T(z) =& S^{-1}(\lambda S(z)) \\ =& S^{-1}(\lambda \frac{z-i}{z+i}) \\ =& \frac{-i\lambda\frac{z-i}{z+i} - i}{\lambda\frac{z-i}{z+i} - 1} \\ =& \frac{(-i\lambda - i)z + (-\lambda + 1)}{(\lambda - 1)z + (-i\lambda - i)} \end{aligned}$$

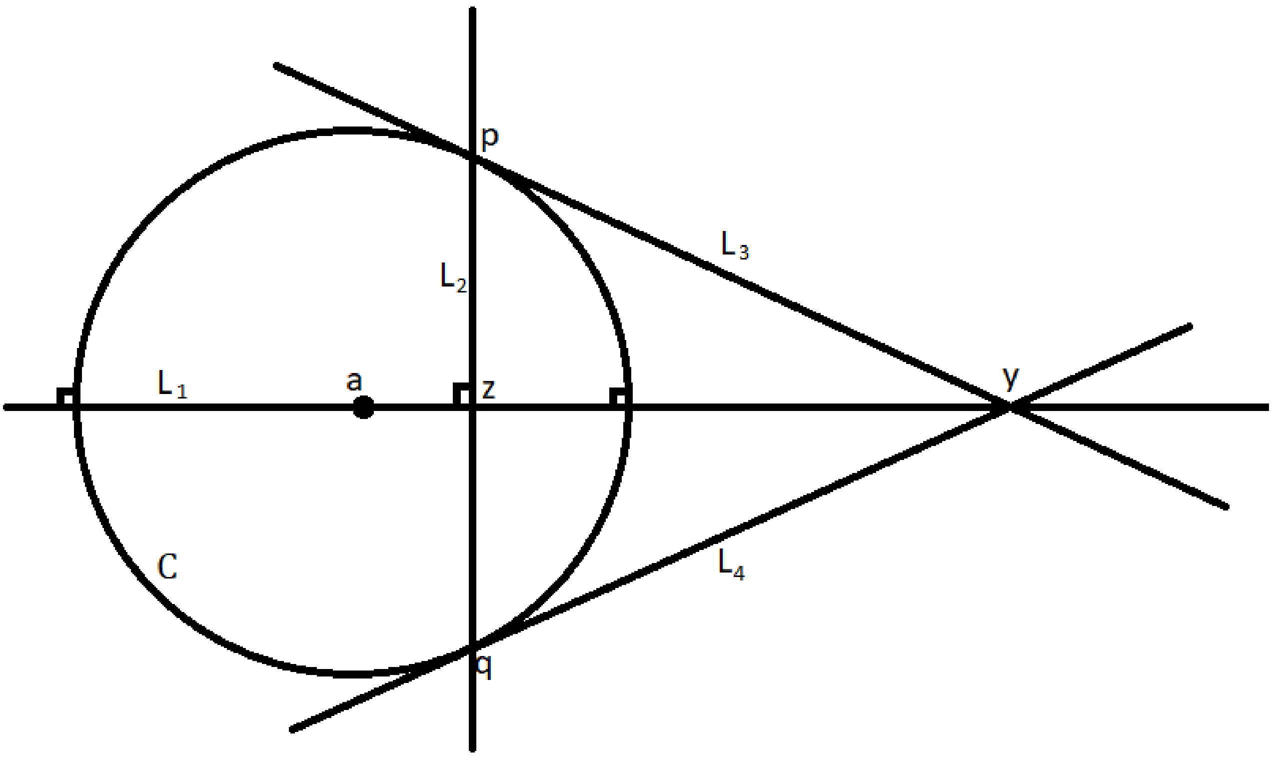
where λ is a non-zero complex number.

4. Let C_1, C_2 be two clines. Fix 3 distinct points on C_1 and 3 distinct points on C_2. By Fundamental Theorem of Möbius geometry, we can find a Möbius transformation T mapping the 3 points on C_1 to the 3 points on C_2. As Möbius transformations map clines to clines, and 3 distinct points uniquely determine a cline, we know that $T(C_1) = C_2$.

5. We will provide 2 solutions for question 5. The first solution uses the formula for symmetric point $|z^* - a||z - a| = R^2$, which is what everybody did. The second solution is purely geometric, and doesn't require any calculations. The second solution shows the beauty of Möbius geometry, and the reason why HK and UK Olympiad Maths teams often use other geometry systems to solve Euclidean geometry problems. Transformations in other geometry systems sometimes make the questions much easier (HK and UK Olympiad Maths teams usually use projective geometry instead of Möbius geometry though). I would recommend you to try to understand the second solution if you want to gain some geometric intuition on Möbius geometry.

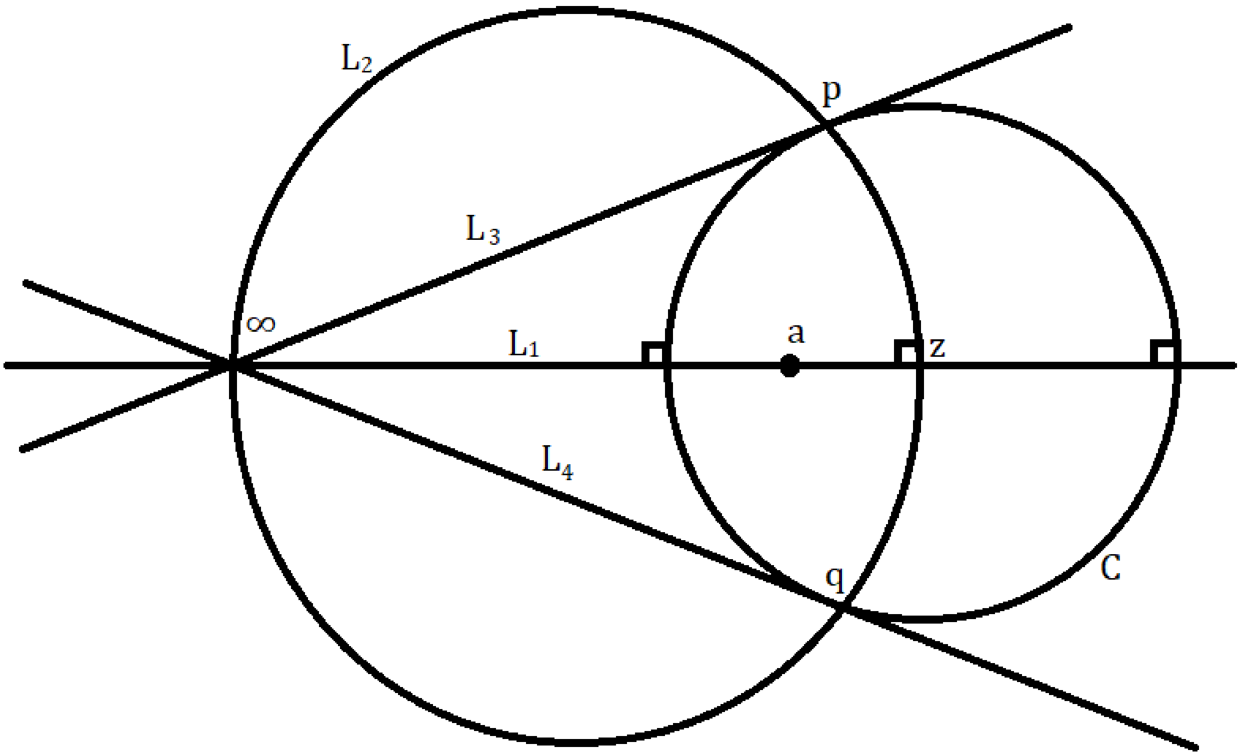
In Mathematics (maybe in real life as well?), situation may become easier when you look at things at a different perspective. That's what we do with the second solution. In Euclidean geometry, sometimes you put a point at $(0,0)$ and rotate some lines so that it's horizontal/vertical, to make the calculations simpler . Similarly, in Möbius geometry, you can apply a Möbius transformation to the figure to make things simpler. To figure out the new figure, you need to use that facts that Möbius transformations preserve angles, map clines to clines, and map symmetric points to symmetric points. It may sounds complicated at the first glance, but it becomes quick and obvious when you gain more and more geometric intuition on Möbius geometry, just like how you do with Euclidean geometry. In the past, personally I find that looking at YouTube videos with beautiful animation on Möbius transformation is a good way to gain geometric intuition.
WARNING: Möbius transformations don't preserve centre of circles.

Solution 1:
Let the tangents to C at p and q meet at y and ∞.



By considering the angles, we can see that triangle zap and triangle pay are similar.
Hence $\frac{|z-a|}{|a-p|} = \frac{|p-a|}{|a-y|}$.
Rearranging the equation, we get $|y-a||z-a| = |p-a|^2 = R^2$. Hence $y = z^*$.

Solution 2:
Re-draw the same picture, with the point y being put at infinity.



In this picture, the angle ∞pz is a right angle as it's an angle at a semi-circle. Hence z is the centre of the circle C. By symmetry, the symmetric point of z in this picture is at infinity. So $y = z^*$.