MATH1010 Revision Exercise 2

1. Compute the first derivative of each of the functions below:

   (a) $\arctan(x+1)$ (b) $\arcsin(x^2)$ (c) $x\arcsin(2x)$ (d) $(\arcsin(x))^2$

   (e) $\dfrac{\arctan(x)}{x}$ (f) $\sqrt{\arctan(2x)}$ (g) $\arctan(\ln(x))$ (h) $\arcsin(\sqrt{1-x^2})$

2. For each of the relations below, find $\dfrac{dy}{dx}$ for the function y implicitly defined by the relation:

   (a) $x = 4y - y^3$ (b) $x = y - \dfrac{1}{y}$ (c) $x = (3y+2)^{10}$ (d) $x = (4-y)(3+y^2)$

   (e) $x = y^{-2}\sin(y)$ (f) $x = \sqrt{\dfrac{y+1}{y+2}}$ (g) $x^2 + y^2 = 4.$ (h) $x^2 + y^2 - 3x + 1 = 0$

   (i) $4y^2 + xy - 6x^2 = 0$ (j) $2x^3 + y^3 - 3x^2y = 1$ (k) $x^2\sin(y) - y\cos(x) = 2$

   (l) $x\cos(y) + y^2\sin(x) = 0$

3. Let n be a positive integer. Let $f(x) = (1-x^2)^n$ for any $x \in \mathbb{R}$.

   (a) Show that $(1-x^2)f'(x) + 2nxf(x) = 0$ for any $x \in \mathbb{R}$.

   (b) Show that $(1-x^2)f^{(n+2)}(x) - 2xf^{(n+1)}(x) + n(n+1)f^{(n)}(x) = 0$ for any $x \in \mathbb{R}$.

4. Let $f(x) = e^x\ln(1+x)$ for any $x \in (-1,+\infty)$.

   (a) Show that $(1+x)f''(x) - (1+2x)f'(x) + xf(x) = 0$ for any $x \in (-1,+\infty)$.

   (b) Let n be a non-negative integer. Show that $(1+x)f^{(n+3)}(x) + (n-2x)f^{(n+2)}(x) + (x-2n-2)f(x) + (n+1)f^{(n)}(x) = 0$ for any $x \in (-1,+\infty)$.

5. Let $f(x) = \dfrac{\ln(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}$ for any $x \in \mathbb{R}$.

   (a) Show that $(1+x^2)f'(x) + xf(x) = 1$ for any $x \in \mathbb{R}$.

   (b) Let n be a non-negative integer. Show that $(1+x^2)f^{(n+2)}(x) + (2n+3)xf^{(n+1)}(x) + (n+1)^2f^{(n)}(x) = 0$ for any $x \in \mathbb{R}$.

6. Let $f(x) = (\arcsin(x))^2$ for any $x \in (-1,1)$.

   (a) Show that $(1-x^2)f''(x) - xf'(x) = 2$ for any $x \in (-1,1)$.

   (b) Let n be a positive integer. Show that $(1-x^2)f^{(n+2)}(x) - (2n+1)xf^{(n+1)}(x) - (n)^2f^{(n)}(x) = 0$ for any $x \in (-1,1)$.

7. Let $f : [3,6] \longrightarrow \mathbb{R}$ be a continuous function. Suppose f is differentiable on $(3,6)$, and $|f'(x) - 9| \leq 3$ on $(3,6)$. Show that $18 \leq f(6) - f(3) \leq 36$.

8. Let $\beta \in (1,+\infty)$. Let $f : (0,+\infty) \longrightarrow \mathbb{R}$ be the function defined by $f(x) = x^\beta + \beta - 1 - \beta x$ for any $x \in (0,+\infty)$.

   (a) i. Compute f'.

      ii. Show that f is strictly decreasing on $(0,1]$.

      iii. Show that f is strictly increasing on $[1,+\infty)$.

      iv. Determine whether f attains the maximum and/or the minimum on $(0,+\infty)$.

   (b) Hence, or otherwise, show that $(1+r)^\beta \geq 1 + \beta r$ for any $r \in (-1,+\infty)$.

9. Prove the following inequalities:

   (a) $\dfrac{x}{1+x^2} < \arctan(x) < x$ for any $x \in (0,+\infty)$.

(b) $0 < \ln(1+x) - \dfrac{2x}{2+x} < \dfrac{x^3}{12}$ for any $x \in (0, +\infty)$.

10. (a) Prove that $1 - \dfrac{x^2}{2} < \cos(x)$ for any $x \in (0, 2\pi]$.

    (b) Prove that $\cos(x) < 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24}$ for any $x \in (0, 2\pi]$.

    (c) Prove that $1 - \dfrac{x^2}{2} < \cos(x) < 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24}$ for any $x \in (2\pi, +\infty)$.

    (d) Prove that $1 - \dfrac{x^2}{2} < \cos(x) < 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24}$ for any $x \in \mathbb{R}\backslash\{0\}$.

11. Apply L'Hôpital's Rule to evaluate each of the limits below.

    (a) $\displaystyle\lim_{x\to 0} \frac{x + \tan(x)}{\sin(2x)}$ (b) $\displaystyle\lim_{x\to 0} \frac{e^x - e^{-x}}{\sin(x)}$ (c) $\displaystyle\lim_{x\to 0} \frac{\arctan(x)}{x}$ (d) $\displaystyle\lim_{x\to 0} \frac{e^{2x} - 1}{2\ln(1+x)}$

    (e) $\displaystyle\lim_{x\to 0} \frac{x^2 + 3x + 4}{3x^3 + 5}$ (f) $\displaystyle\lim_{x\to 0} \frac{x - \sin(x)}{2x^3}$ (g) $\displaystyle\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin(x)}$ (h) $\displaystyle\lim_{x\to 1} \frac{e^{x-1} - x}{(x-1)^2}$

    (i) $\displaystyle\lim_{x\to 0} \frac{24\cos(x) - 24 - 12x^2 + x^4}{\sin^6(x)}$ (j) $\displaystyle\lim_{x\to 0} \frac{x\tan(x)}{1 - \sqrt{1 - x^2}}$ (k) $\displaystyle\lim_{x\to +\infty} \frac{\ln(e^x + x^2)}{x^2}$

    (l) $\displaystyle\lim_{x\to 1} \frac{1 + \ln(x) - x^x}{1 + \ln(x) - x}$ (m) $\displaystyle\lim_{x\to 0^+} \frac{(\ln(x))^5}{\sqrt[5]{x}}$ (n) $\displaystyle\lim_{x\to +\infty} \frac{\ln(1 + xe^{2x})}{\sin^2(x)}$

    (m) $\displaystyle\lim_{x\to 0^+} \frac{\ln(\sin(\alpha x))}{\ln(\sin(\beta x))}$. (Here α, β are positive real numbers.)

12. Evaluate each of the limits below. When necessary, apply L'Hôpital's Rule.

    (a) $\displaystyle\lim_{x\to 0^+} x^2 e^{(-x^{-2})}$ (b) $\displaystyle\lim_{x\to 0^+} \sin(x)\ln(x)$ (c) $\displaystyle\lim_{x\to 0^+} x\csc(2x)$ (d) $\displaystyle\lim_{x\to +\infty} x\left[\left(1 + \frac{1}{x}\right)^x - e\right]$

13. Evaluate each of the limits below. When necessary, apply L'Hôpital's Rule.

    (e) $\displaystyle\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\arctan(x)}\right)$ (f) $\displaystyle\lim_{x\to 0^+} \left(\csc^2(x) - \frac{1}{x^2}\right)$

    (g) $\displaystyle\lim_{x\to 1^+} \left(\frac{x^2}{(1-x)^2} - \frac{1}{(\ln(x))^2}\right)$ (h) $\displaystyle\lim_{x\to 0^+} \left(\cot(x) - \frac{1}{x}\right)$

14. Evaluate each of the limits below. When necessary, apply L'Hôpital's Rule.

    (a) $\displaystyle\lim_{x\to +\infty} x^{\frac{1}{x}}$ (b) $\displaystyle\lim_{x\to 0^+} x^{\sin(x)}$ (c) $\displaystyle\lim_{x\to 0^+} \left(\ln\left(\frac{1}{x}\right)\right)^x$ (d) $\displaystyle\lim_{x\to +\infty} \left(1 - \frac{1}{x}\right)^{-x}$

    (e) $\displaystyle\lim_{x\to +\infty} \left(1 + \frac{2}{x}\right)^{-x}$ (f) $\displaystyle\lim_{x\to +\infty} \left(1 + \frac{3}{x^2}\right)^x$ (g) $\displaystyle\lim_{x\to +\infty} \left(\frac{x+1}{x-1}\right)^x$ (h) $\displaystyle\lim_{x\to +\infty} \left(\frac{x^2+1}{x^2-1}\right)^{x^2}$

    (i) $\displaystyle\lim_{x\to +\infty} \left(\frac{x^2 - 2x - 3}{x^2 - 3x - 28}\right)^x$ (j) $\displaystyle\lim_{x\to \frac{\pi}{2}^-} (\tan(x))^{\cos(x)}$ (k) $\displaystyle\lim_{x\to 0^+} (1 + \sin(x))^{\frac{1}{x}}$

    (l) $\displaystyle\lim_{x\to 0^+} (1 + \sin^2(x))^{\frac{1}{x}}$ (m) $\displaystyle\lim_{x\to 1^+} x^{\frac{e^x}{1-x}}$ (n) $\displaystyle\lim_{x\to 0^+} (1 - \cos(x))^{\frac{1}{\ln(x)}}$

    (o) $\displaystyle\lim_{x\to \frac{\pi}{2}^-} (\cos(x))^{\ln(\sin(x))}$ (p) $\displaystyle\lim_{x\to 0} \left(\frac{\arcsin(x)}{x}\right)^{\frac{1}{x^2}}$ (q) $\displaystyle\lim_{x\to 0} \left(\frac{\sin(x)}{x}\right)^{\frac{1}{x^2}}$

15. Evaluate the each of the limits below. Think carefully whether to apply L'Hôpital's Rule or not.

    (a) $\displaystyle\lim_{x\to +\infty} \frac{x + \sin(x)}{x - \sin(x)}$ (b) $\displaystyle\lim_{x\to +\infty} \frac{e^x + x\sin(x) + \cos(x)}{e^x + \cos(x)}$ (c) $\displaystyle\lim_{x\to +\infty} \frac{x^2 + \sin(2x)}{(2x^3 + x + \sin(x))e^{\sin(x)}}$