MATH 1010E Notes

Week 11

Topics covered

- Riemann Sum
- Fundamental theorem of calculus
- Applications

Until now, when we talked about integral, we mean "indefinite integral" or the solutions to the differential equation $F'(x) = f(x)$.

We have denoted such integrals by the symbol $\int f(x)dx$.

We also noticed that $\int f(x)dx$ and $\int f(x)dx + C$ are both solutions to the differential equation $F'(x) = f(x)$.

But "integration" has another meaning. It is the "computation" of "area" under the curve $y = f(x), a \leq x \leq b$.

Q: How to define this kind of integral? What is its name?
A: It is called definite integral and is defined as follows:

Suppose we have a continuous function $f: [a, b] \to \mathbb{R}$ and we want to compute the "area" under the curve $y = f(x)$, for $x \in [a, b]$. The we can do this by the following method:

(Step 1) Partition the interval $[a, b]$ into n subintervals defined by the points

$$a = x_0 < x_1 < \cdots < x_{i-1} < x_i < \cdots < x_n = b$$

This way, we have n subintervals, i.e. $[x_0, x_1], [x_1, x_2], \cdots, [x_{i-1}, x_i], \cdots, [x_{n-1}, x_n]$.

(Step 2) Define the symbol $\llbracket P \rrbracket$ (you can call it "length" of P) by letting

$$\llbracket P \rrbracket = maximum\ among\ x_1 - x_0, x_2 - x_1, \cdots, x_i - x_{i-1}, \cdots, x_n - x_{n-1}$$

Therefore, if $\llbracket P \rrbracket \to 0$, then all the numbers $x_1 - x_0, x_2 - x_1, \cdots, x_i - x_{i-1}, \cdots, x_n - x_{n-1}$ will go to zero.

(Step 3) Construct n rectangles "under" the curve $y = f(x)$, by choosing as heights the numbers $f(\xi_i)$, where ξ_i is any number between x_{i-1} and x_i. Choose widths to be the numbers $x_i - x_{i-1}$.

Such rectangles have then areas equal to $f(\xi_i) \cdot (x_i - x_{i-1})$
The sum of these areas is then equal to

$$\sum_{i=1}^{n} f(\xi_i) \cdot (x_i - x_{i-1})$$

or equal to

$$\sum_{i=1}^{n} f(\xi_i) \cdot \Delta x_i$$

if we let $\Delta x_i = x_i - x_{i-1}$.

(Step 4) Now one can show (with more mathematics) that for continuous function f, as $\|P\| \to 0$, the following limit is always a finite number:

$$\lim_{\|P\| \to 0} \sum_{i=1}^{n} f(\xi_i) \cdot \Delta x_i$$

(Step 5) Finally, we give a symbol to this limit and call it $\int_a^b f(x)dx$.

In conclusion, we have (for continuous function $f: [a, b] \to \mathbb{R}$) the following:

$$\lim_{\|P\| \to 0} \sum_{i=1}^{n} f(\xi_i) \cdot \Delta x_i = \int_a^b f(x)dx.$$

Remark: This kind of sum are called Riemann sums.

This limit, $\int_a^b f(x)dx$ is called the "definite integral" of f for $a \leq x \leq b$.

Example
Consider the function $f(x) = x$, for $0 \leq x \leq 1$.

Partition $[0,1]$ into n subinterval of the form:

$$\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \cdots, \left[\frac{i-1}{n}, \frac{i}{n}\right], \cdots, \left[\frac{n-1}{n}, \frac{n}{n}\right]$$

Each of these subintervals has length $\frac{1}{n}$, therefore $\|P\| = \frac{1}{n}$, which means as $\|P\| = \frac{1}{n} \to 0$, it follows that $n \to \infty$.

Next, consider the following sum of areas of rectangles, where we choose $\xi_i = x_i = \frac{i}{n}$, then we have the sum

$$\sum_{i=1}^{n} f(x_i) \cdot \Delta x_i = \sum_{i=1}^{n} f\left(\frac{i}{n}\right) \cdot \frac{1}{n} = \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{1}{n} = \sum_{i=1}^{n} \frac{i}{n^2}$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} i = \frac{1}{n^2} \cdot \frac{(1+n)n}{2} = \frac{n+1}{2n} = \left(\frac{1}{2}\right)\left(1 + \frac{1}{n}\right)$$

Hence, as $\|P\| \to 0$, it follows that $n \to \infty$ and also $\lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i) \cdot \Delta x_i = \left(\frac{1}{2}\right) \lim_{n \to \infty} \left(1 + \frac{1}{n}\right) = \frac{1}{2}$.

Remark: The choice of the points ξ_i is arbitrary. One can choose (i) the left endpoint, (ii) the right endpoint, (iii) the mid-points, (iv) the absolute maximum points, (v) the absolute minimum points etc.

No matter what one chooses for ξ_i, the limit remains the same.

Properties of Definite Integrals

The following properties of definite integrals are consequences of the area of a rectangle.

1. $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
2. $\int_a^b kf(x)dx = k \int_a^b f(x)dx$
3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
4. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

One also has the following simple inequality (which hasn't been mentioned in the lectures),

5. If $f(x) \leq g(x)$, $a \leq x \leq b$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.

Remark: Using the above-mentioned Riemann Sum method to find area under a curve $y = f(x), a \leq x \leq b$ is very tedious. There is a more effective method, which computes area by (i) first compute an indefinite integral $F(x) = \int f(x)dx + C$, then (ii) compute the number $F(b) - F(a)$. This number is the the area wanted.

This method is called the Fundamental Theorem of Calculus.

Remark: This method doesn't always work. For some functions, such as $f(x) = e^{x^2}$, one cannot find a "closed form" function $F(x) = \int e^{x^2} dx + C$. For such functions $f(x)$, the areas have to computed using other methods, such as the Riemann sum.

Fundamental Theorem of Calculus

There are two parts in the Fundamental Theorem of Calculus (in the future, we just write "FTC" for it).

(Part I)

Let $f(x)$ be a continuous function defined on the closed interval $[a, b]$. Then the following holds

$$\frac{d \int_a^x f(t)dt}{dx} = f(x)$$

for each $x \in (a, b)$.

(**Terminology:** We call this function $\int_a^x f(t)dt$ the "area-finding function". This function computes the area "under" the curve $y = f(t)$ for those t from a to x.)

(Part II)

For any solution $F(x)$ which satisfies the "differential" equation

$$F'(x) = f(x) \text{ for } x \in (a, b),$$

we can compute the area under the curve $y = f(x)$ for $a \leq x \leq b$, by

$\int_a^b f(t)dt = F(b) - F(a)$

Note that one can use any symbol, e.g. x, u instead of t here. I.e.

$$\int_{x=a}^{x=b} f(x)dx = \int_{u=a}^{u=b} f(u)du = \int_{t=a}^{t=b} f(t)dt = F(b) - F(a)$$

Some ideas of the Proof of Part I and Part II

(Part I)

(Step 1) We prove that $A(x) = \int_a^x f(t)dt$ is differentiable for any $x \in (a, b)$.

To do this, we (as always) first consider the difference quotient, i.e.

$$\frac{A(x+h) - A(x)}{h} = \frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h}$$

But we know (from properties of Definite integrals) that:

$$\frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h} = \frac{\int_x^{x+h} f(t)dt}{h}$$

(Step 2)

Next, using the Mean Value Theorem for Integrals, we obtain

$$\int_x^{x+h} f(t)dt = f(\xi) \cdot h$$

where ξ is between x and $x + h$.

(Step 3)

Conclusion: Dividing through by h $(h \neq 0)$, we obtain

$$\frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h} = \frac{\int_x^{x+h} f(t)dt}{h} = \frac{f(\xi) \cdot h}{h} = f(\xi)$$

(Step 4)

Finally, let $h \to 0$, and we obtain

$$\lim_{h \to 0} \frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h} = \lim_{h \to 0} f(\xi) = f(x)$$

The last equality, i.e. $= f(x)$, is true because "as $h \to 0$, by Sandwich Theorem, $\xi \to h$."

Final Conclusion:

We have proved

$$\lim_{h\to 0}\frac{A(x+h)-A(x)}{h}=f(x)$$

I.e. $A'(x)=f(x)$.

(Part II)

We want to prove "For any indefinite integral, i.e. solution $F(x)$, of the differential equation $F'(x)=f(x) - - - - (*)$", one has

$$\int_a^b f(t)dt = F(b)-F(a)$$

Proof:

(Step 1)

We need to use the following theorem, which we mentioned before:

> **Theorem**
>
> Let $F_1(x), F_2(x)$ be any two solutions of the differential equation (*), i.e.
>
> $$F_1'(x)=f(x)$$
>
> and
>
> $$F_2'(x)=f(x)$$
>
> for all $x\in(a,b)$. Then $F_1(x)-F_2(x)=C$ for all $x\in(a,b)$.
>
> (In short, it says "any two indefinite integrals of $f(x)$ differ only by a constant".)

Q: How to use the Theorem in the box above?

A: We let $F_1(x)=A(x)$ and $F_2(x)=F(x)$, where $F(x)$ is any solution of $F'(x)=f(x)$, $a<x<b$. Then by the theorem in the box, we have

$$A(x)-F(x)=C, \qquad a<x<b$$

But then we have two cases.

(Step 2)

The case $x=a$.

In this case, $A(a)=0$, so we get from the above formula that $A(a)-F(a)=C$, which leads to the conclusion that $F(a)=-C$.

(Step 3)

The case $x=b$.

In this case, $A(b)=\int_a^b f(x)dx$, so $A(b)-F(b)=\int_a^b f(x)dx-F(b)=C$.

But remembering that in Step 2, we have obtained $C = F(a)$. Putting this into the formula $A(b) - F(b) = -F(a)$, gives $A(b) = F(b) - F(a)$.

Further F.T.C.

One can greatly extend the FTC to compute things like the following:

$$\frac{d}{dx}\int_{t=a(x)}^{t=b(x)} f(x,t)dt$$

Goal: We want to show that (in the following, for simplicity, we omit write $t = a(x), t = b(x)$).

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)dt = f\big(x,b(x)\big)b'(x) - f\big(x,a(x)\big)a'(x) + \int_{a(x)}^{b(x)} \frac{\partial f(x,t)}{\partial x}dt$$

Proof:

(Main Idea): Instead of $\int_{a(x)}^{b(x)} f(x,t)dt$, we consider the, more general, expression

$\int_A^B f(C,t)dt$ and think of it as a "function of 3 variables A, B and C).

(Step 1) For a function of several variables, say 3 variables, we have the following Chain Rule (if $A = a(x), B = b(x), C = c(x)$):

$\frac{df(A,B,C)}{dx} = \frac{df(a(x),b(x),c(x))}{dx} = f_A\big(a(x),b(x),c(x)\big)\cdot\frac{da(x)}{dx} + f_B\big(a(x),b(x),c(x)\big)\cdot\frac{db(x)}{dx} + f_C\big(a(x),b(x),c(x)\big)\cdot\frac{dc(x)}{dx}$

where $\frac{\partial f}{\partial A} = \lim_{h\to 0}\frac{f(A+h,B,C)-f(A,B,C)}{h}$, $\frac{\partial f}{\partial B} = \lim_{k\to 0}\frac{f(A,B+k,C)-f(A,B,C)}{k}$, $\frac{\partial f}{\partial C} = \lim_{l\to 0}\frac{f(A,B,C+l)-f(A,B,C)}{l}$

(i.e. differentiating ONLY with respect to the 1st variable, respectively the 2nd or the 3rd variable.

Shorter notation: $\frac{\partial f}{\partial A}\Big|_{(a(x),b(x),c(x))} = f_A\big(a(x),b(x),c(x)\big)$. Similar for $f_B\big(a(x),b(x),c(x)\big)$, $f_C\big(a(x),b(x),c(x)\big)$)

An Example

If f is a function of 3 variables, A, B, C and each of these variables depends on x. Then f is a function of x. The Chain Rule then says

$$\frac{\partial f}{\partial x} = f_A \cdot \frac{dA}{dx} + f_B \cdot \frac{dB}{dx} + f_C \cdot \frac{dC}{dx}$$

Example:

$$f(A,B,C) = A + B^2 + BC$$

Suppose $A = \cos x, B = \sin x, C = x$ Then

$$\frac{df}{dx} = \frac{\partial f}{\partial A} \cdot \frac{d\cos x}{dx} + \frac{\partial f}{\partial B} \cdot \frac{d\sin x}{dx} + \frac{\partial f}{\partial C} \cdot \frac{dx}{dx}$$

But now $\frac{\partial f}{\partial A} = 1$, (because now *B,C* are constants)
$\frac{\partial f}{\partial B} = 2B + C, \frac{\partial f}{\partial C} = B$

Putting these back into the formula $\frac{df}{dx} = \frac{\partial f}{\partial A} \cdot \frac{d\cos x}{dx} + \frac{\partial f}{\partial B} \cdot \frac{d\sin x}{dx} + \frac{\partial f}{\partial C} \cdot \frac{dx}{dx}$
we get $\frac{df}{dx} = -\sin x + (2B + C)\cos x + B = -\sin x + (2\sin x + x)\cos x + \sin x$
We can check that the computation is correct by the following direct computation:

$$f = \cos x + \sin^2 x + (\sin x)x$$

$$\frac{df}{dx} = -\sin x + 2\sin x\cos x + (\cos x)x + \sin x$$

(Step 2)

Apply the Chain Rule to the following function, F, of 3 variables:

$F(A,B,C) = \int_A^B f(C,t)dt$ and get

$$\begin{aligned}\frac{dF}{dx} &= \frac{\partial \int_A^B f(C,t)dt}{\partial A} \cdot \frac{da(x)}{dx} + \frac{\partial \int_A^B f(C,t)dt}{\partial B} \cdot \frac{db(x)}{dx} + \frac{\partial \int_A^B f(C,t)dt}{\partial C} \cdot \frac{dc(x)}{dx} \\ &= \frac{\partial - \int_B^A f(C,t)dt}{\partial A} \cdot \frac{da(x)}{dx} + \frac{\partial \int_A^B f(C,t)dt}{\partial B} \cdot \frac{db(x)}{dx} + \frac{\partial \int_A^B f(C,t)dt}{\partial C} \cdot \frac{dc(x)}{dx} \\ &= -f(C,A) \cdot \frac{da(x)}{dx} + f(C,B) \cdot \frac{db(x)}{dx} + \int_A^B \frac{\partial f(C,t)}{\partial C} dt \cdot \frac{dc(x)}{dx} \\ &= f(C,B) \cdot \frac{db(x)}{dx} - f(C,A) \cdot \frac{da(x)}{dx} + \int_A^B \frac{\partial f(C,t)}{\partial C} dt \cdot \frac{dx}{dx}\end{aligned}$$

Because $A = a(x), B = b(x), c(x) = x$, finally, we obtain

$$\frac{d\int_{a(x)}^{b(x)} f(x,t)dt}{dx} = f(x,b(x)) \cdot \frac{db(x)}{dx} - f(x,a(x)) \cdot \frac{da(x)}{dx} + \int_{a(x)}^{b(x)} \frac{\partial f(x,t)}{\partial x} dt$$