1. assume $\sup(S)$ is not unique.

let S_1, S_2 be two different no. and both of them are $\sup(S)$.

by def, $\begin{cases} \forall x \in S, & x \le S_i \\ \forall \varepsilon > 0, \ \exists y_i \in S, & \text{s.t } y_i > S_i - \varepsilon \end{cases}$ $i = 1, 2$

$\therefore \quad S_1 - \varepsilon < y_1 \le S_2$

as ε is arbitrary, $\therefore \quad S_1 \le S_2$.

similarly, $\quad S_2 - \varepsilon < y_2 \le S_1$

$\Rightarrow \quad S_2 \le S_1$

$\therefore \quad S_1 = S_2$ lead to contradiction.

2. by def, $\begin{cases} \forall x \in A, & x \le \sup(A) \\ \forall \varepsilon > 0, & \exists y \in A \ \text{s.t} \ y > \sup(A) - \varepsilon \end{cases}$

$\forall y \in bA, \quad \exists \hat{x} \in A.$

s.t $\quad y = b\hat{x}$

$\Rightarrow \quad y \le b \sup(A)$ as b is positive.

$\therefore \quad b \sup(A)$ is upper bd of bA.

$\forall \delta > 0, \ \exists \hat{y} \in A$ s.t $\quad \hat{y} > \sup(A) - \delta/b$

$\Rightarrow \quad b\hat{y} > b \sup(A) - \delta$

as $\quad b\hat{y} \in bA$

$\therefore \quad b \sup(A)$ satisfy the second cond.

$\therefore \quad b \sup(A) = \sup(bA)$

3. $\forall\ x \in (a,b) \quad a < x < b$

$\therefore$ b is upper bd of (a,b)

$\forall\ \varepsilon > 0, \quad b - \min\{\frac{b-a}{2}, \varepsilon/2\} \in (a,b)$

and

$$b - \min\{\frac{b-a}{2}, \varepsilon/2\} \geq b - \varepsilon/2$$

$$> b - \varepsilon$$

$\therefore$ b satisfy the 2nd cond.

$\therefore$ $\sup(a,b) = b$.

4a. unbound above and below.

$\forall\ M > 0$

$\forall\ \cancel{x} \ 0 < x < \min\{\frac{1}{M}, 1\}$

$\frac{1}{x^5} > \frac{1}{x}$ as $x^4 < 1$ and x is positive.

$> M \quad \therefore$ unbded above.

$\forall\ 0 > x > -\min\{\frac{1}{M}, 1\}$

$\frac{1}{x^5} < \frac{1}{x}$ as x is neg and $x^4 < 1$

$< -M$

$\therefore$ unbded below.

4b. unbded above and below.

similar to 4a.

5 bded above and below.

$\forall n \in \mathbb{N}, \quad \frac{2^n - 1}{2^n + 1} < \frac{2^n}{2^n + 1} < \frac{2^n}{2^n} = 1 \quad \therefore$ S bded above by 1

$\frac{2^n - 1}{2^n + 1} \geq 0$ as $2^n \geq 1$

$\therefore$ S bded below by 0

6a. $\forall M > 1$

$\forall x > (2M)^{1/100}$ then $\frac{1}{2}x^{101} > \frac{1}{2}(2M)^{\frac{101}{100}} > 1$

$$\frac{x^{101}-1}{x-1} > \frac{x^{101}-1}{x} > \frac{x^{101}-\frac{1}{2}x^{101}}{x} = \frac{1}{2}x^{100}$$

$$> M$$

$$\therefore \lim_{x\to\infty} \frac{x^{101}-1}{x-1} = \infty$$

6b.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$\forall M > 0$

$\forall x > M$

$\forall k \in \mathbb{N}$ $\quad \frac{x^k}{k!} > 0$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$> 1 + x$$

$$> x > M$$

$$\therefore \lim_{x\to\infty} e^x = +\infty$$

6c

$$\frac{d}{dx}(1-x^2)^2 = -4x(1-x^2)$$

$(1-(1)^2)^2 = 0$ is the only turning pt in $(0,2)$

if $x=0$, $(1-0^2)^2 = 1$

if $x=2$, $(1-2^2)^2 = 9$

$\therefore \sup S = 9 \qquad \inf S = 0$