**2055 Notes 1**

**Topics**
**Upper Bounds, Lower Bounds, Sup, Inf**

Let $A$ be a subset of the real line, i.e. $\mathbb{R}$. We say '$A$ is bounded (from) above' if there exists a number $M$ such that

$$a \in A \implies a \leq M$$

In other words, $A$ is bounded above if there is a number $M$ greater than (or equal to) all elements of $A$. This number will be called an upper bound of $A$.

Comments:

(1) If $A$ has an upper bound $M$, one can easily find other upper bounds, such as $M+1, M+2, \cdots$

(2) An upper bound of a set, e.g. $A$, may or may not belong to the set $A$.

  (a) Let $A = \{1, 2\}$ be a subset of the set of natural numbers $\mathbb{N}$, then $2, 3, 4, 5, \cdots$ will all be upper bounds of $A$, but only 2 belongs to $A$.

  (b) Let $A = \{x \mid 0 < x < 1\}$ be subset of rational numbers $\mathbb{Q}$, then $1, 1+\frac{1}{2}, 1+\frac{3}{4}, \cdots$ are all upper bounds of $A$. However, none of them belongs to $A$.

**Terminology**
If an upper bound, say $M$, of a set $A$ belongs to the set $A$, then we say $M$ is a maximum of $A$.

Comments:

(1) Similar to upper bound, we can define lower bound.

(2) If a set $A$ is both bounded above and bounded below, then we say $A$ is bounded.

A set which has no upper bounds is called unbounded set.

To say it mathematically, we write

**Def.**
Let $A$ be a subset of $\mathbb{R}$. It does not admit any upper bound if for any (given) $M$, there exists an element $a$ (in $A$) such that $a > M$. Such a

set is called a set which is unbounded above.

In symbols, we write

$A$ is unbounded above if

$$\forall M \in \mathbb{R} \; \exists a \in A \mid a > M.$$

Comment:
In this definition, $a$ depends on the given $M$. To emphasize this fact, we sometimes write $a_M$.

Here below is a simple result, which demonstrates how one proves things concerning upper and lower bounds.

**Lemma**
Let $A$ and $B$ be two subsets of $\mathbb{R}$. If both of them are bounded above, then their union $A \cup B$ is also bounded above. (Similar result holds if one changes the word 'bounded above' to 'bounded below')

**Proof.**
Let $M_1$ be an upper bound of $A$, $M_2$ be an upper bound of $B$.

(This means: If $a \in A$ then $a \leq M_1$; If $a \in B$ then $a \leq M_2$.)

Now take any $a$ in $A \cup B$. Then $a \in A$ or $a \in B$, implying
$a \leq M_1$ or $a \leq M_2$.

(This motivates us to consider the larger of these two numbers, $M_1$ and $M_2$ !)

Let's denote this number by $M \overset{\text{def}}{=} \max\{M_1, M_2\}$, then we get $a \leq M_1 \leq M$ or $a \leq M_2 \leq M$.

In both cases, the $a$ is less than $M$ ! (Hence we have found an upper bound for $A \cup B$) $\square$

**Unboundedness of $N$**

The set of natural numbers, denoted by the symbol $\mathbb{N}$, i.e the set

$$\{0, 1, 2, 3 \cdots\}$$

is a set which is bounded below but not bounded above.

This can be described by the sentence:

| |
|---|
| for each given <u>real</u> no. $r$ there exists a <u>natural no.</u> $n$ such that $n > r$ |

In symbols, this can be written as:

$$\forall r \in \mathbb{R}\ \exists n \in \mathbb{N} \mid n > r$$

<u>Comment:</u>
Since $n$ depends on $r$, we may write it as $n_r$ (instead of $n$) if we want to emphasize this dependence!

This property is known as the <u>Archimedean Property</u>, which can also be rewritten in the following (equivalent) form:

$$\forall \epsilon > 0\ \exists n \in \{1, 2, 3, \cdots\} \mid \frac{1}{n} < \epsilon$$

**Supremum, Infimum**
Now we come to two very important concepts about the real numbers, namely 'supremum' and 'infimum'.

Let $A$ be a subset of $\mathbb{R}$. Suppose that $A$ is bounded above. Then we know that 'among all the upper bounds of $A$, there is at most one which <u>belongs to</u> $A$. If this happens, we say that this upper bound is a <u>maximum</u> of $A$'.

**E.g.**
$(0, 1)$ is a subset of $\mathbb{R}$. Then the set of all upper bounds of $A$ is the set

$$B = [1, \infty)$$

But none of the elements in $B$ is in $A$.

$$\boxed{\text{none of the upper bounds is a maximum of } A}$$

**E.g.**
$(0, 1]$ is a subset of $\mathbb{R}$. The set of all upper bounds of $A$ is the set

$$B = [1, \infty)$$

and $1 \in A$. $\boxed{\text{1 is the maximum of } A}$

**Supremum**
Since $A$ is bounded above, the set

$$B = \{y \in \mathbb{R} \mid y \text{ is an upper bound of } A\}$$

is a non-empty set. The most important difference between rational numbers and real numbers is that this set $B$ has always a minimum (i.e. in the set itself).

**Property of the real number line**
Suppose $A$ is bounded above. Then the set of all upper bounds of $A$ admits a minimum in $\mathbb{R}$.

Comment: This property is discovered by the German mathematician, Dedekind, so it is also called the Dedekind property.

Now we can define supremum of $A$ by
**Def.**
Let $A$ be a subset of $\mathbb{R}$. Then the minimum (if it exists!) of the upper bounds of $A$ is called $\sup(A)$ ('supremum of $A$').

**Characterization of sup($A$)**
Now we can rewrite mathematically what the definition of supremum describes:

If we denote the supremum of $A$ by $L$, i.e. $L = \sup(A)$, then since $L$ is the 'least among the upper bounds of $A$', we have

- ($L$ is a upper bound of $A$) i.e. $a \in A \implies a \leq L$
  (More precisely, you can write $\forall a \; | a \in A \implies a \leq L$)
- ($L$ is the least among upper bounds) i.e.
  $\forall \epsilon > 0, \exists a \in A \mid a > L - \epsilon$

Comments:

(1) 'supremum' is also known as 'least upper bound'. We can either write $\sup(A)$ for it or $\mathrm{lub}(A)$ for it.
(2) The second bullet point above means 'if we subtract any small positive quantity from $L$, $L$ will no longer be upper bound of $A$'.

(3) In the Definition for supremum, we have not assumed that $A$ is bounded above. So there are two cases:
  - $A$ is bounded above, then $\sup(A)$ exists,
  - $A$ is unbounded above, then we denote 'unboundedness' by $\sup(A) = \infty$.

(4) In a similar way, we can define for $A$ bounded from below the 'greatest lower bound' or 'infimum' with the notation $\inf(A)$ (or $\mathrm{glb}(A)$)

**E.g.**
$A = \{1 - \frac{1}{n} \mid n$ is a non-zero natural number$\}$, then $\sup(A) = 1$ (but 1 is not an element of $A$!)

**E.g.**
$A = \{\frac{1}{n} \mid n$ is a non-zero nat. no.$\}$, then $\inf(A) = 0$

**Application of Dedekind Property**
Historically, one important use of Dedekind property is the construction of 'real nos.' from the 'rational numbers'.

We will not go into this too much, but to get a feeling for this, just consider the equation

$$b = \sqrt{2}$$

Question is: How can you 'define' (i.e. 'talk about') $\sqrt{2}$ if you are only allowed to use rational numbers?

(Note that this equation can be rewritten in the form $b^2 = 2$, which has no square root sign!)

The following points are important:

(1) If we work in $\mathbb{R}$ (i.e. we are allowed to use <u>real</u> nos., then this equation <u>has a solution</u>, given by

$$b = \sup\{x \mid b^2 \leq 2\},$$

because of the Dedekind property.

(2) If we work in $\mathbb{Q}$ (i.e. we can only use rational nos.), then the <u>supremum</u> doesn't exist (because it is the irrational number $\sqrt{2}$)

(3) To <u>build</u> real nos. from rational numbers, one consider sets of the form
$$A = \{x \mid b^2 \leq 2\}$$
and
$$B = \{x \mid b^2 \geq 2\}$$
and use them to define $\sqrt{2}$. (Notice that in the description of set $A$ and set $B$, we do not need to mention 'square root').

<u>Comments:</u>

(1) Instead of talking about '$\sqrt{2}$', one talks about the pair of subsets $A$ and $B$ (both subsets of $\mathbb{Q}$) and use them to 'define' $\sqrt{2}$. This method is known as 'Dedekind cut'.
(2) In a similar way to defining $\sqrt{2}$, one can use other pairs of subsets similar to $A$ and $B$ to define any real number.