MATH2550 I

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Webpage: (in progress) www.math.cuhk.edu.hk/course/1819/math2550

Introduction. There are three main topics in this course, they are: Multivariable calculus, Ordinary Differential Equations & Matrices.

We will first talk about Multivariable calculus.

What is it? Before going into that, let's rethink what calculus is about.

Calculus is about "functions". Typically one studies (i) derivative of a function, they are written symbolically as something such as $\frac{df}{dx}$, or $\left.\frac{df}{dx}\right|_{x=a}$ or $f'(x)$ or $f'(a)$, (ii) integrals of a function. Symbolically, they are $\int f(x)dx$ or $\int_a^b f(x)dx$.

Remarks.

- Meaning of $f'(a)$ is as follows – this number is the "slope" of the "tangent" line to the curve $y = f(x)$ at the point given by $x = a, y = f(a)$.
- One important point is this: For a straight line e.g. $y = f(x) = mx + c$, we have one and only one slope. This slope is given by rise/run and mathematically by the formula $\frac{f(a+h)-f(a)}{h}$ which is equal to m. (Conclusion: For straight line, the choice of the numbers a and h are irrelevant. No matter what a, h are, the slope is still the same number m).
- If $f(x)$ has any other form than the form $mx + c$, then the ratio $\frac{f(a+h)-f(a)}{h}$ will depend on (i) a and also on (ii) h. (Conclusion: there are many such ratios; we have to find a good "representative" out of these ratios. One way to do it is to consider the limit of such ratios, written as $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$) If such a limit exists (it may sometimes not exist!) then we denote this limit by the symbol $f'(a)$. Hence we get the formula $f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$.
- The above number is the "slope of the tangent line to the curve $y = f(x)$ as the point $x = a, y = f(a)$).

An Example. Consider the function given by $f(x) = x^2$ and a point $x = a$ $(f(a) = a^2)$, then following our recipe, we have $f'(a) = \lim_{h\to 0} \frac{(a+h)^2 - a^2}{h}$

$$= \lim_{h\to 0} \frac{a^2+2ah+h^2-a^2}{h} = \lim_{h\to 0} \frac{2ah+h^2}{h} = \lim_{h\to 0}(2a+h) = \lim_{h\to 0} 2a + \lim_{h\to 0} h = 2a + 0 = 2a.$$

Now we can define "partial derivative" of a function of 2 variables x and y. Suppose we have a function depending on x & y, written in the form $f(x,y)$ or $z = f(x,y)$.

Copying the idea from before, we can define the "parital derivative" of $f(x,y)$ with respect to the variable x (at the point $x = a, y = b, z = f(a,b)$) by

$\lim_{h\to 0} \frac{f(a+h,y) - f(a,b)}{h}$ and give it the symbol $f_x(a,b)$.

Other notations. One can also write it as $\left.\frac{\partial f}{\partial x}\right|_{x=a,y=b}$ or $\left.\frac{\partial f}{\partial x}\right|_{(a,b)}$

Similarly, one can define $\left.\frac{\partial f}{\partial y}\right|_{x=a,y=b}$ by $\lim_{k\to 0} \frac{f(a,y+k) - f(a,b)}{k}$

How to compute $f_x(a,b), f_y(a,b)$.
You just imagine one of the variables is constant and differentiate w.r.t. x or y.

Example. $f(x,y) = x^2 + y^2 + 1$, $x = a, y = b$. Then $f_x(a,b) = 2a, f_y(a,b) = 2b$
Example. $f(x,y) = 2x^2 + 3y^2 + 1$, $x = a, y = b$. Then $f_x(a,b) = 4a, f_y(a,b) = 6b$
Example. $f(x,y) = e^{xy}\sin(xy)$ and we ignore writing a, b for simplicity. Then $f_x =$

$$\frac{\partial e^{\overset{u}{\widetilde{xy}}}}{\partial u} \cdot \sin(xy) + e^{xy} \frac{\partial \sin(xy)}{\partial x} = \frac{de^u}{du} \cdot \frac{\partial u}{\partial x} \cdot \sin(xy) + e^{xy} \frac{\partial \sin(xy)}{\partial x}$$
$$= e^{xy} \cdot y \cdot \sin(xy) + e^{xy}\cos(xy)\, y$$

Remark. For simplicity, we have ignored the variables x, y when we wrote f_x (i.e. we could have written $f_x(x,y)$ but this is too complicated!)

Higher partial derivatives

Having computed f_x, f_y we can go on to differentiate their partial derivatives w.r.t.

x or y and get $\frac{\partial f_x}{\partial x}$ or $\frac{\partial f_y}{\partial x}$ or $\frac{\partial f_x}{\partial y}$

Notations. We write $\frac{\partial f_x}{\partial x}$ as $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$, $\frac{\partial^2 f}{\partial x^2}$ or f_{xx}

Similarly we have $\frac{\partial f_y}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$

Finally we have $\frac{\partial f_y}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{xy}$

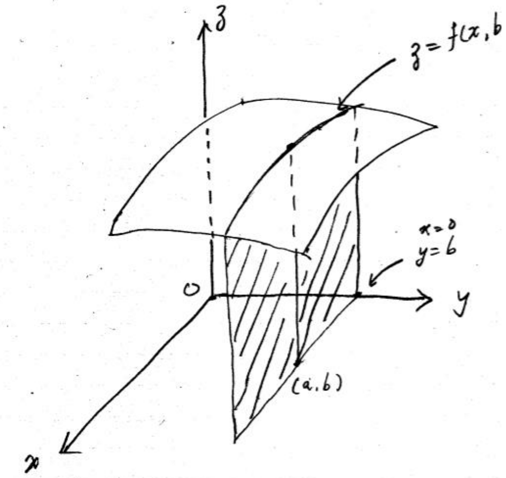
Remark. One an show that for "nice" functions, it holds that $f_{xy} = f_{yx}$.

Example. Let $f(x, y) = \ln\sqrt{x^2 + y^2}$, then $f_{xx} + f_{yy} = 0$ if $(x, y) \neq (0,0)$.

Geometric Meaning of $f_x(a, b), f_y(a, b)$.

- First of all, $z = f(x, y)$ represents a "surface" in the 3D space (like $y = f(x)$ represents a curve in the 2D plane);
- Given the point $x = a, y = b$, we can erect a vertical plane (parallel to the xz −plane) containing this point;
- This vertical plane cuts the surface $z = f(x, y)$ along a curve;
- This curve has a tangent line above the point $x = a, y = b$;
- The slope of this tangent line is $f_x(a, b)$

- If we change the plane in bullet point 2 to a vertical plane (parallel to the yz −plane), we get another curve. The slope to this curve above the point $x = a, y = b$ is the number $f_y(a, b)$.



Gradient Opertor/Vector

Example (from School Math). We need the following thing: (*) A curve "represented" in the form $f(x,y) = c$. A good example is this:

$$x^2 + y^2 = R^2$$

It is a circle centered at the origin with radius R.

Then we can (ii) compute the following vector (**) $\hat{\imath}\, f_x + \hat{\jmath}\, f_y$, (for simplicity, we have written f_x instead of $f_x(x,y)$, but the meaning is the same). Here the expression $\hat{\imath}$ means "going 1 unit in the direction of the x −axis". More mathematically, one can say $\hat{\imath} = (1,0)$. Similarly, $\hat{\jmath} = (0,1)$ (i.e. going 1 unit along the $y-$ axis direction).

Notation/Terminology. We give the name "gradient operator" to the object

$$\hat{\imath}\,\frac{\partial}{\partial x} + \hat{\jmath}\,\frac{\partial}{\partial y}$$

(A commonly used notation for this is ∇, i.e. $\nabla = \hat{\imath}\,\frac{\partial}{\partial x} + \hat{\jmath}\,\frac{\partial}{\partial y}$)

This operator, when it encounters a function (of 2 variables), yields the result

$$\hat{\imath}\,\frac{\partial f}{\partial x} + \hat{\jmath}\,\frac{\partial f}{\partial y}$$

(known as "gradient vector")

Or more concisely, $\hat{\imath}\, f_x + \hat{\jmath}\, f_y$ (or if you prefer using "coordinates", the expression (f_x, f_y).

Important Remark.

The "gradient vector" points in a direction perpendicular to the curve $f(x,y) = c$.

Let's see that this is correct via our example.

In our example, $f_x = 2x, f_y = 2y$ and so the gradient vector is $\nabla f = \hat{\imath}\,(2x) + \hat{\jmath}\,(2y)$ (or you can write $2x, 2y$ to the left hand side of $\hat{\imath}, \hat{\jmath}$).

The 3D case.

For the 3D case, the phenomenon is the same. Now you have a function of 3 variables, say x, y, z and you consider a "surface" given by the following form

$$f(x,y,z) = c$$

Then the gradient operator is $\nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

where $\hat{\imath} = (1,0,0), \hat{\jmath} = (0,1,0), \hat{k} = (0,0,1)$.

Applying this operator to a function, e.g. f, gives a vector perpendicular to the surface.

Example.

Consider the sphere of radius R centered at the origin, in the form

$$x^2 + y^2 + z^2 = R^2$$

Then $\nabla f = \hat{\imath}\frac{\partial f}{\partial x} + \hat{\jmath}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z} = 2x\hat{\imath} + 2y\,\hat{\jmath} + 2z\,\widehat{k}$

is a vector perpendicular to this sphere (Note that this vector starts at the point (x, y, z) on the sphere!)