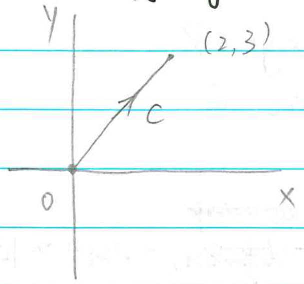
1. Find $\int_C (x+y)\,ds$, where C is the straight line joining $(0,0)$ to $(2,3)$



$\vec{v} = (2,3) - (0,0) = (2,3)$

① $\alpha : [0,1] \longrightarrow \mathbb{R}^2$ defined as

$$\alpha(t) = (x(t), y(t)) = (0,0) + t\cdot(2,3) = (2t, 3t)$$

$$\alpha'(t) = (2,3) \Rightarrow |\alpha'(t)| = \sqrt{4+9} = \sqrt{13}$$

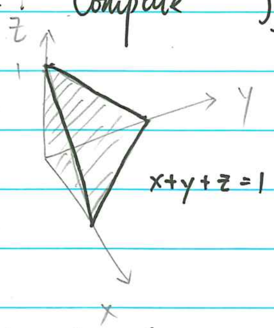
<u>Note</u> : $\alpha(0) = (0,0)$, $\alpha(1) = (2,3)$

(You can try $\tilde{\alpha} : [0,2] \longrightarrow \mathbb{R}^2$,
$\tilde{\alpha}(t) = (t, \frac{3}{2}t)$,
then $\tilde{\alpha}(0) = (0,0)$, $\tilde{\alpha}(2) = (2,3)$)

② $$\int_C (x+y)\,ds = \int_0^1 (2t+3t)\,|\alpha'(t)|\,dt = \int_0^1 5t\cdot\sqrt{13}\,dt = \frac{5\sqrt{13}}{2}$$

Try $\tilde{\alpha}$:
$$\int_0^2 \left(t+\tfrac{3}{2}t\right)\cdot|\tilde{\alpha}'(t)|\,dt = \int_0^2 \tfrac{5}{2}t\cdot\left|\left(1,\tfrac{3}{2}\right)\right|\,dt$$
$$= \int_0^2 \tfrac{5}{2}t\cdot\sqrt{1+\tfrac{9}{4}}\,dt = \int_0^2 \tfrac{5\sqrt{13}}{4}\,t\,dt = \frac{5\sqrt{13}}{2}$$

2. Compute $\iiint_D 1\,dz\,dy\,dx$, where D is the triangular prism



<u>Key point</u> : Determine the upper & lower limit of the integral.

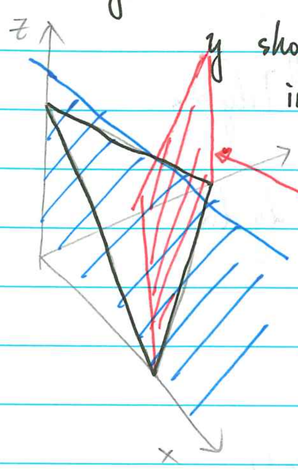
① Consider z :

Any point in D lies above the x-y plane $\Rightarrow z \geq 0$

It also lies below $x+y+z=1 \Rightarrow z \leq 1-x-y$.

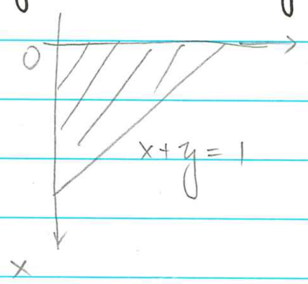
So the inner most one is $\int_0^{1-x-y} 1\, dz$

② Consider y : (Determine the range of y, at this stage, y should be independent of z as after integration in ①, it should get rid of z !)



(Bounded by two planes)

Or you can directly consider the 2D picture



$\Rightarrow \quad 0 \le y \le 1-x$

③ Consider x : (the range of x is $[0,1]$)

$\Rightarrow \quad 0 \le x \le 1$.

Hence $\iiint_D 1\, dz\, dy\, dx = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 1\, dz\, dy\, dx$

$$= \int_0^1 \int_0^{1-x} 1-x-y\, dy\, dx$$

$$= \int_0^1 \left[(1-x)^2 - \frac{y^2}{2} \Big|_0^{1-x} \right] dx$$

$$= \int_0^1 (1-x)^2 - \frac{(1-x)^2}{2} + \frac{0}{2}\, dx$$

$$= \int_0^1 \frac{(1-x)^2}{2}\, dx$$

$$= -\frac{(1-x)^3}{6} \Big|_0^1 = \frac{1}{6}$$

<u>Remark</u> : $\iiint_D 1\, dx\, dy\, dz = \text{Volume of } D$

3. Compute $\iiint_D (1-x-y-z)\,dz\,dy\,dx$, D is the same as Q2.

$$\iiint_D (1-x-y-z)\,dz\,dy\,dx = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (1-x-y-z)\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \left[(1-x-y)^2 - \left(\frac{z^2}{2}\right)\Big|_0^{1-x-y} \right] dy\,dx$$

$$= \int_0^1 \int_0^{1-x} (1-x-y)^2 - \frac{(1-x-y)^2}{2}\, dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \frac{(1-x-y)^2}{2}\, dy\,dx$$

$$= \int_0^1 - \frac{(1-x-y)^3}{6}\Big|_0^{1-x} dx$$

$$= \int_0^1 \frac{(1-x)^3}{6}\,dx = -\frac{(1-x)^4}{24}\Big|_0^1 = \frac{1}{24}.$$