MATH2550

Course Work 2

1. Compute the following determinant $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$.

2. Compute the following determinant $\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$. What do you notice when you compare the result of Q1 and Q2?

3. Compute the following determinant $\begin{vmatrix} a+b & b & c \\ d+e & e & f \\ g+h & h & i \end{vmatrix}$.

4. Repeat Q3 by replacing $+$ with a $-$.

5. Compute the following determinant $\begin{vmatrix} b & a & c \\ e & d & f \\ h & g & i \end{vmatrix}$. What do you notice?

   Does the same conclusion follow when we interchange any two columns in a determinant?

6. In Q5, what happens if we interchange two rows of a determinant?

7. Compute the following determinant $\begin{vmatrix} \lambda a & b & c \\ \lambda d & e & f \\ \lambda g & h & i \end{vmatrix}$.

8. What happens in Q7, if we multiply each entry in column 2 by the same number λ (pronounced as "lambda").

9. Show using School Geometry, that $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is the area of the parallelogram made out of the two vector $a\,\hat{\imath} + b\,\hat{\jmath}$ and $c\,\hat{\imath} + d\,\hat{\jmath}$.

1. $$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$
$$= a(ei-hf) - b(di-gf) + c(dh-ge) \qquad (*)$$

2. $$\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} = a\begin{vmatrix} e & h \\ f & i \end{vmatrix} - d\begin{vmatrix} b & h \\ c & i \end{vmatrix} + g\begin{vmatrix} b & e \\ c & f \end{vmatrix}$$
$$= a(ei-fh) - d(bi-ch) + g(bf-ce)$$
$$= a(ei-fh) + b(-di+gf) + c(dh-ge) = (*)$$

3. $$\begin{vmatrix} a+b & b & c \\ d+e & e & f \\ g+h & h & i \end{vmatrix} = (a+b)\begin{vmatrix} e & f \\ h & i \end{vmatrix} - (d+e)\begin{vmatrix} b & c \\ h & i \end{vmatrix} + (g+h)\begin{vmatrix} b & c \\ e & f \end{vmatrix}$$
$$= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - d\begin{vmatrix} b & c \\ h & i \end{vmatrix} + g\begin{vmatrix} b & c \\ e & f \end{vmatrix}$$
$$+ b\begin{vmatrix} e & f \\ h & i \end{vmatrix} \cancel{a} - e\begin{vmatrix} b & c \\ h & i \end{vmatrix} + h\begin{vmatrix} b & c \\ e & f \end{vmatrix}$$
$$= \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} b & b & c \\ e & e & f \\ h & h & i \end{vmatrix}$$
$$= \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (*)$$

$$\left(\begin{vmatrix} b & b & c \\ e & e & f \\ h & h & i \end{vmatrix} = b(ei-hf) - b(ei-hf) + c(eh-eh) = 0+0 = 0 \right)$$

4. By the same calculation in 3.,
$$\begin{vmatrix} a-b & b & c \\ d-e & e & f \\ g-h & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} - \begin{vmatrix} b & b & c \\ e & e & f \\ h & h & i \end{vmatrix}$$
$$= \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (*)$$

5. $$\begin{vmatrix} b & a & c \\ e & d & f \\ h & g & i \end{vmatrix} = b(di-gf) - a(ei-hf) + c(eg-hd)$$
$$= a(hf-ei) - b(gf-di) + c(eg-hd) \qquad (\#)$$

<u>Note</u> : $$(*) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a(ei-hf) - b(-gf+di) + c(dh-ge) = (-1)\cdot(\#)$$

Yes to the second question. (Leave as an exercise :))

6. Switching row 1 and 2:

$$\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix} \overset{Q1+Q2}{=} \begin{vmatrix} d & a & g \\ e & b & h \\ f & c & i \end{vmatrix} \overset{Q5}{=} - \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$$

$$\overset{Q1+Q2}{=} - \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

7.

$$\begin{vmatrix} \lambda a & b & c \\ \lambda d & e & f \\ \lambda g & h & i \end{vmatrix} = \lambda a(ei-hf) - \lambda d(bi-ch) + \lambda g(bf-ce)$$

$$= \lambda \cdot \left[a(ei-hf) - d(bi-ch) + g(bf-ce) \right] \quad (\Delta)$$

Note: $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a(ei-hf) - d(bi-ch) + g(bf-ce)$

$$(\Delta) = \lambda \cdot \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

8. (I'm not sure about the question, so I'll do both cases)

① $$\begin{vmatrix} \lambda a & \lambda b & c \\ \lambda d & \lambda e & f \\ \lambda g & \lambda h & i \end{vmatrix} \overset{Q7}{=} \lambda \cdot \begin{vmatrix} a & \lambda b & c \\ d & \lambda e & f \\ g & \lambda h & i \end{vmatrix} \overset{Q5}{=} (-1) \cdot \lambda \begin{vmatrix} \lambda b & a & c \\ \lambda e & d & f \\ \lambda h & g & i \end{vmatrix}$$

$$\overset{Q7}{=} (-1) \cdot \lambda^2 \begin{vmatrix} b & a & c \\ e & d & f \\ h & g & i \end{vmatrix} \overset{Q5}{=} (-1)^2 \cdot \lambda^2 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

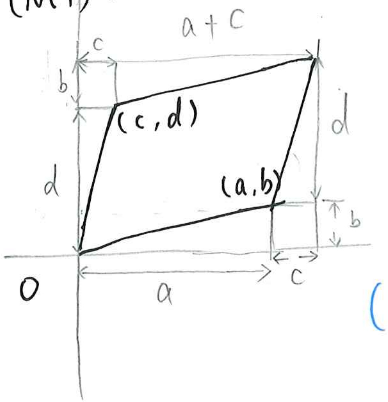
$$= \lambda^2 \cdot \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

② $$\begin{vmatrix} a & \lambda b & c \\ d & \lambda e & f \\ g & \lambda h & i \end{vmatrix} = - \begin{vmatrix} \lambda b & a & c \\ \lambda e & d & f \\ \lambda h & g & i \end{vmatrix} = -\lambda \begin{vmatrix} b & a & c \\ e & d & f \\ h & g & i \end{vmatrix}$$

$$= \lambda \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

(M1):

9.



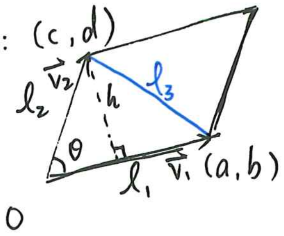
$$\text{Area}(\square) = (a+c)\cdot(b+d) - 2bc - \tfrac{1}{2}\cdot c \cdot d - \tfrac{1}{2} a \cdot b - \tfrac{1}{2} d \cdot c - \tfrac{1}{2} a \cdot b$$

$$= \cancel{ab} + ad + bc + \cancel{cd} - 2bc - \cancel{cd} - \cancel{ab}$$

$$= ad - bc$$

(Put $\square$ into a large rectangle)

(M 2):



$$\text{Area}(\square) = l_1 \cdot h = l_1 \cdot l_2 \cdot \sin\theta$$

$$l_1 = \sqrt{a^2 + b^2}$$

$$l_2 = \sqrt{c^2 + d^2}$$

$$\sin\theta = \sqrt{1 - \cos^2\theta} \quad (\theta \in [0, \pi) \Rightarrow \sin\theta > 0)$$

By cosine law, $\cos\theta = \dfrac{l_1^2 + l_2^2 - l_3^2}{2 l_1 l_2}$

$$l_3 = |\vec{v_1} - \vec{v_2}| = \sqrt{(a-c)^2 + (b-d)^2}$$

$$\Rightarrow \cos\theta = \frac{(a^2+b^2) + (c^2+d^2) - [(a-c)^2 + (b-d)^2]}{2 l_1 \cdot l_2}$$

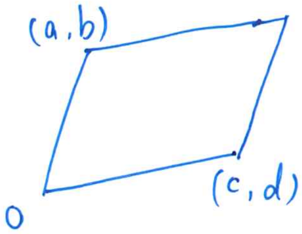
$$= \frac{(a^2+b^2) + (c^2+d^2) - a^2 + 2ac - c^2 - b^2 + 2bd - d^2}{2 l_1 l_2} = \frac{ac + bd}{l_1 \cdot l_2}$$

$$\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \frac{(ac+bd)^2}{l_1^2 \cdot l_2^2}} = \sqrt{\frac{a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 - (a^2c^2 + 2abcd + b^2d^2)}{(a^2+b^2)(c^2+d^2)}}$$

$$= \frac{\sqrt{a^2d^2 - 2abcd + b^2c^2}}{l_1 \cdot l_2} = \frac{\sqrt{(ad-bc)^2}}{l_1 \cdot l_2} = \frac{|ad - bc|}{l_1 \cdot l_2}$$

So $\text{Area}(\square) = l_1 \cdot l_2 \cdot \sin\theta = |ad - bc|$

<u>Remark</u>: Absolute value is needed, since it may be the case that



Then by M1, $\text{Area}(\square) = cb - ad$

(Depends on the ~~orig~~ orientation of ~~(a, b) × (c, d)~~ $(a\hat{i} + b\hat{j}) \times (c\hat{i} + d\hat{j})$)

But area is always nonnegative, so $|\cdot|$ is needed.