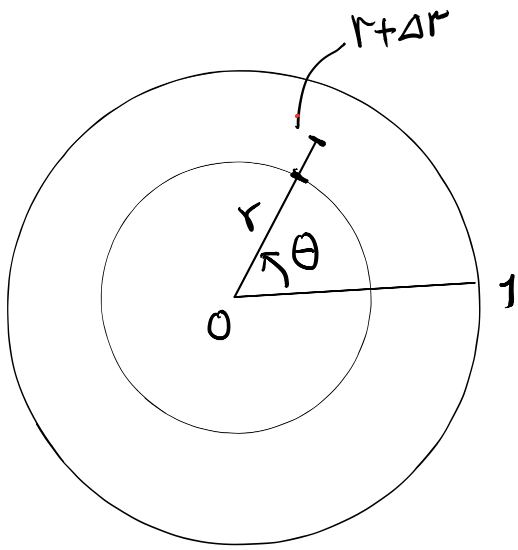
Area in the Disk Model



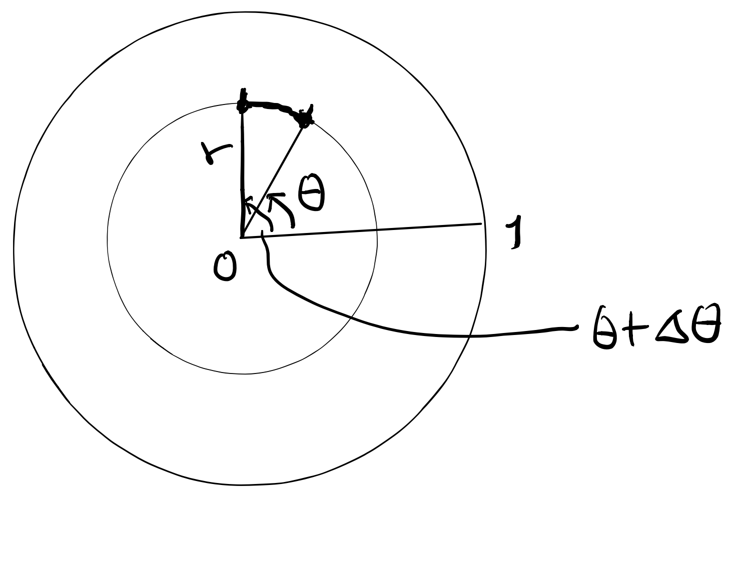
We first calculate the length elements for $\theta = \text{const.}$ and $r = \text{const.}$ in the disk model.

$z(r) = re^{i\theta}$, θ = fixed, r is the parameter.

$\Rightarrow z'(r) = e^{i\theta}$

$$\text{length} = \int_r^{r+\Delta r} \frac{2|z'(r)|}{1-|z(r)|^2}\, dr$$

$$\sim \frac{2}{1-r^2}\Delta r$$

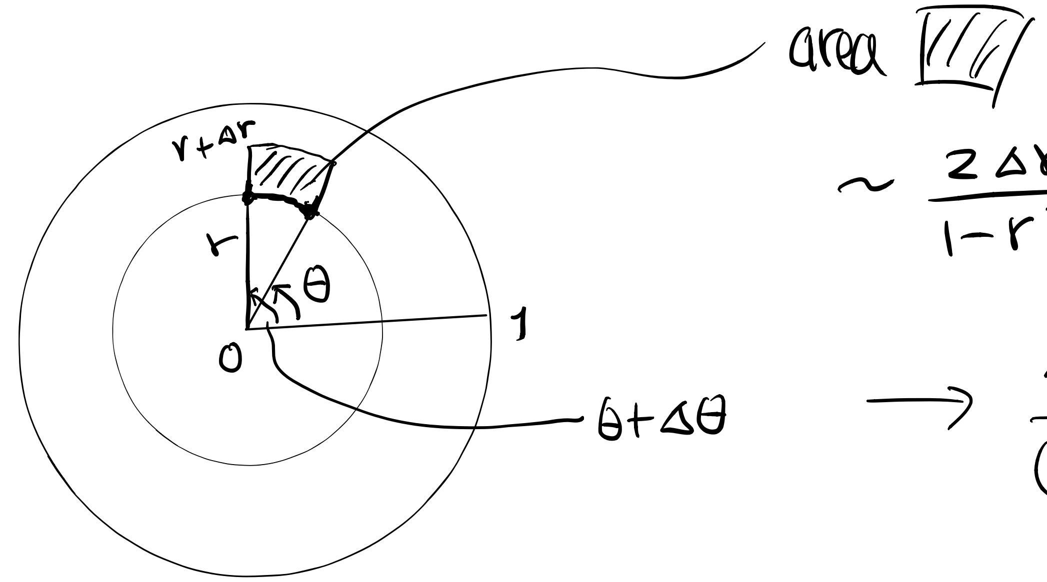


$z(\theta) = re^{i\theta}$, r = fixed, θ is the parameter.

$z'(\theta) = ire^{i\theta}$

$$\text{length} = \int_\theta^{\theta+\Delta\theta} \frac{2|z'(\theta)|}{1-|z(\theta)|^2}\, d\theta$$

$$\sim \frac{2r}{1-r^2}\Delta\theta$$



$$\sim \frac{2\Delta r}{1-r^2}\cdot\frac{2r}{1-r^2}\Delta\theta$$

$$\longrightarrow \frac{4r\,dr\,d\theta}{(1-r^2)^2}$$

Def: The <u>area</u> of a region R in the hyperbolic plane (unit disk model) is defined by

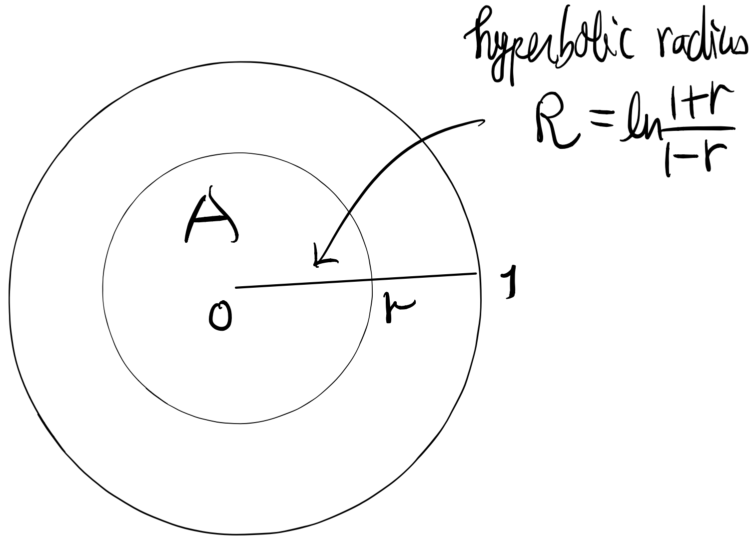
$$\boxed{A = \iint_R \frac{4r}{(1-r^2)^2}\,dr\,d\theta}$$

$$= \iint_R \frac{4}{(1-x^2-y^2)^2}\,dx\,dy$$

eg: Area enclosed by a hyperbolic circle of hyperbolic radius R is $4\pi\sinh^2\left(\frac{R}{2}\right)$.

Pf: $A = \int_0^{2\pi}\int_0^{r} \frac{4r}{(1-r^2)^2}\,dr\,d\theta$ $\left(\text{where } \sinh x = \frac{e^x - e^{-x}}{2}\right)$

$$= 4\pi \int_0^r \frac{2r\,dr}{(1-r^2)^2}$$



$$= 4\pi \int_1^{1-r^2} \frac{-dz}{z^2}$$

(substitute $z = 1-r^2$)

$$= 4\pi \left[\frac{1}{z}\right]_1^{1-r^2}$$

$$= 4\pi \left(\frac{1}{1-r^2} - 1\right)$$

$$= 4\pi \frac{r^2}{1-r^2}$$

Since $R = \ln \frac{1+r}{1-r}$,

$$r = \frac{e^R - 1}{e^R + 1}$$

Hence

$$A = 4\pi \frac{\left(\frac{e^R-1}{e^R+1}\right)^2}{1 - \left(\frac{e^R-1}{e^R+1}\right)^2}$$

$$= 4\pi \frac{(e^R-1)^2}{(e^R+1)^2 - (e^R-1)^2}$$

$$= 4\pi \cdot \frac{(e^R - 1)^2}{4e^R}$$

$$= 4\pi \cdot \left(\frac{e^R - 1}{2e^{\frac{R}{2}}} \right)^2$$

$$= 4\pi \left(\frac{e^{\frac{R}{2}} - e^{-\frac{R}{2}}}{2} \right)^2$$

$$= 4\pi \left(\sinh \frac{R}{2} \right)^2$$

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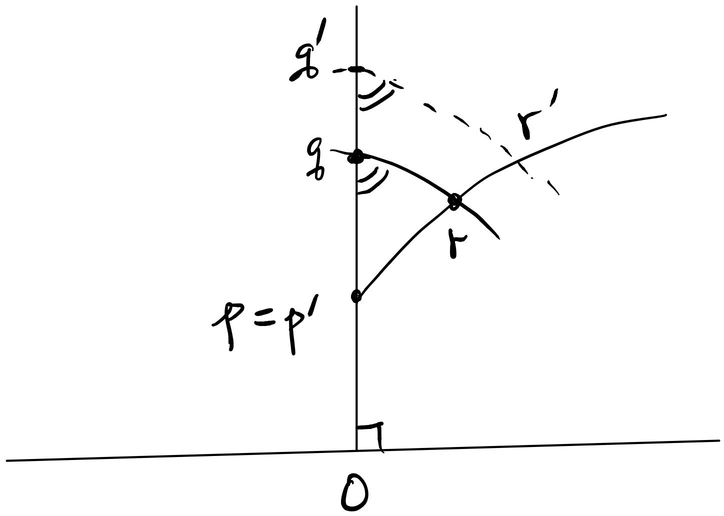
Similarity

Thm: If corresponding angles are equal in 2 (hyperbolic) triangles Δpqr & $\Delta p'q'r'$, then the hyperbolic triangles are congruent.

Proof: In upper half-plane model, we may put $p = p'$ and $\overline{pq}$ & $\overline{p'q'}$ along the y-axis and both q, q' above p.



If $q \neq q'$, by a scaling, which is a transformation in the hyperbolic group, the hyperbolic straight line containing $\overline{qr}$ transforms to a hyperbolic straight line passing through the point q', which makes an angle equal to

$\angle pqr = \angle p'q'r'$

$\Rightarrow$ (the pt.) r' is on this hyperbolic straight line (by assumption)

$\Rightarrow$ r' is the intersection point of this hyperbolic straight line and $\overline{pr}$.

This implies $A(\Delta pqr) \neq A(\Delta p'q'r')$ which is a contradiction since both areas equal to

$$\pi - (\text{sum of interior angles}).$$

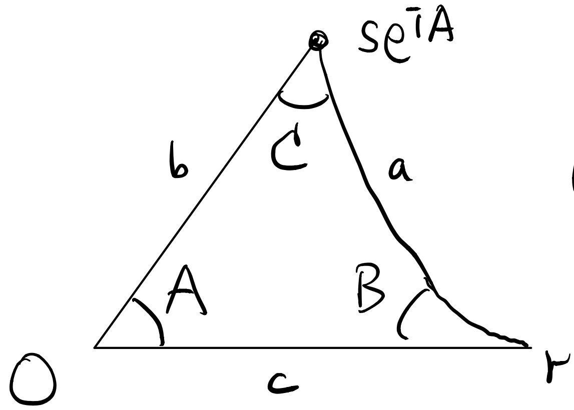
$\therefore\ q = q'$, ie. $\overline{pq} \cong \overline{p'q'}$

Similarly $\overline{pr} \cong \overline{p'r'}$ & $\overline{qr} \cong \overline{q'r'}$.

Hence $\Delta pqr \cong \Delta p'q'r'$. ✗✗

Cosine rule I (in hyperbolic geometry)

disk model



$(0 < r, s < 1)$

Then

$$\boxed{\operatorname{ch} a = \operatorname{ch} b \operatorname{ch} c - \operatorname{sh} b \operatorname{sh} c \cos A}$$

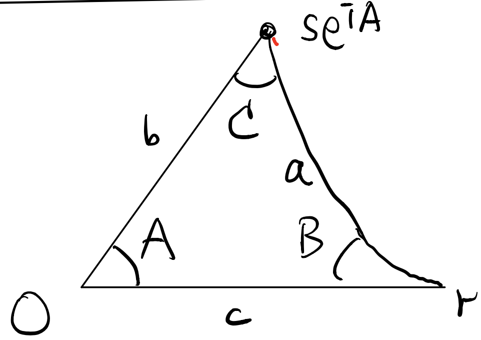
where
$$\begin{cases} \operatorname{ch} x = \cosh x = \dfrac{e^x + e^{-x}}{2} \\ \operatorname{sh} x = \sinh x = \dfrac{e^x - e^{-x}}{2} \end{cases}$$

Cosine Rule II

$$\boxed{\cosh a = \frac{\cos B \cos C + \cos A}{\sin B \sin C}}$$

Sine Rule

$$\boxed{\frac{\sin A}{\sinh a} = \frac{\sin B}{\sinh b} = \frac{\sin C}{\sinh c}}$$



Pf of Cosine Rule I

By our notation, we have

$$|r - se^{iA}|^2 = r^2 + s^2 - 2rs\cos A$$

In hyperbolic geometry

$$\begin{cases} c = \ln \frac{1+r}{1-r}, \quad b = \ln \frac{1+s}{1-s}, \text{ and} \\ a = \ln \dfrac{1 + \left|\frac{r - se^{iA}}{1 - rse^{iA}}\right|}{1 - \left|\frac{r - se^{iA}}{1 - rse^{iA}}\right|} \end{cases}$$

$$\Rightarrow \quad r = \frac{e^c - 1}{e^c + 1} = \frac{\cancel{e^{\frac{c}{2}}}(e^{\frac{c}{2}} - e^{-\frac{c}{2}})/2}{\cancel{e^{\frac{c}{2}}}(e^{\frac{c}{2}} + e^{-\frac{c}{2}})/2} = \frac{sh\frac{c}{2}}{ch\frac{c}{2}} = \tanh\frac{c}{2}$$

Similarly

$$\begin{cases} r = \tanh\frac{c}{2} \\ s = \tanh\frac{b}{2} \\ \left|\frac{r - se^{iA}}{1 - rse^{iA}}\right| = \tanh\frac{a}{2} \end{cases}$$

$$\Rightarrow \quad \tanh^2\frac{a}{2} = \frac{|r - se^{iA}|^2}{|1 - rse^{iA}|^2} = \frac{r^2 - 2rs\cos A + s^2}{1 - 2rs\cos A + r^2s^2}$$

$$\Rightarrow \quad ch\,a = \frac{ch\,a}{1} = \frac{ch^2\frac{a}{2} + sh^2\frac{a}{2}}{ch^2\frac{a}{2} - sh^2\frac{a}{2}} \quad (\text{Ex!})$$

$$= \frac{1 + \tanh^2\frac{a}{2}}{1 - \tanh^2\frac{a}{2}}$$

$$= \frac{(1 - 2rs\cos A + r^2s^2) + (r^2 - 2rs\cos A + s^2)}{(1 - 2rs\cos A + r^2s^2) - (r^2 - 2rs\cos A + s^2)}$$

$$= \frac{1 + r^2 + s^2 + r^2s^2 - 4rs\cos A}{1 - r^2 - s^2 + r^2s^2}$$

$$= \frac{(1+r^2)(1+s^2) - 4rs\cos A}{(1-r^2)(1-s^2)}$$

$$= \left(\frac{1+r^2}{1-r^2}\right)\left(\frac{1+s^2}{1-s^2}\right) - \left(\frac{2r}{1-r^2}\right)\left(\frac{2s}{1-s^2}\right)\cos A$$

$$\frac{1+r^2}{1-r^2} = \frac{1+\tanh^2\frac{c}{2}}{1-\tanh^2\frac{c}{2}} = \frac{\mathrm{ch}^2\frac{c}{2} + \mathrm{sh}^2\frac{c}{2}}{\mathrm{ch}^2\frac{c}{2} - \mathrm{sh}^2\frac{c}{2}} = \frac{\mathrm{ch}\,c}{1} = \mathrm{ch}\,c$$

Similarly $\frac{1+s^2}{1-s^2} = \mathrm{ch}\,b$

$$\frac{2r}{1-r^2} = \frac{2\tanh\frac{c}{2}}{1-\tanh^2\frac{c}{2}} = \frac{\frac{2\,\mathrm{sh}\frac{c}{2}}{\mathrm{ch}\frac{c}{2}}}{1-\frac{\mathrm{sh}^2\frac{c}{2}}{\mathrm{ch}^2\frac{c}{2}}}$$

$$= \frac{2\,\mathrm{sh}\frac{c}{2}\,\mathrm{ch}\frac{c}{2}}{\mathrm{ch}^2\frac{c}{2} - \mathrm{sh}^2\frac{c}{2}} = \frac{\mathrm{sh}\,c}{1} = \mathrm{sh}\,c \quad (\text{Ex!})$$

Similarly $\frac{2s}{1-s^2} = \mathrm{sh}\,b$.

Hence $\mathrm{ch}\,a = \mathrm{ch}\,b\,\mathrm{ch}\,c - \mathrm{sh}\,b\,\mathrm{sh}\,c\cos A$. #

(Pf of Cosine Rule II is omitted)

Pf of Sine Rule

by Cosine Rule I,

$$\left(\frac{\sin A}{\operatorname{sh} a}\right)^2 = \frac{1-\cos^2 A}{\operatorname{sh}^2 a}$$

$$= \frac{1-\left(\frac{\operatorname{ch} b \operatorname{ch} c - \operatorname{ch} a}{\operatorname{sh} b \operatorname{sh} c}\right)^2}{\operatorname{sh}^2 a}$$

$$= \frac{\operatorname{sh}^2 b \operatorname{sh}^2 c - (\operatorname{ch} b \operatorname{ch} c - \operatorname{ch} a)^2}{\operatorname{sh}^2 a \operatorname{sh}^2 b \operatorname{sh}^2 c}$$

$$= \frac{(\operatorname{ch}^2 b - 1)(\operatorname{ch}^2 c - 1) - (\operatorname{ch}^2 b \operatorname{ch}^2 c - 2\operatorname{ch} a \operatorname{ch} b \operatorname{ch} c + \operatorname{ch}^2 a)}{\operatorname{sh}^2 a \operatorname{sh}^2 b \operatorname{sh}^2 c}$$

$$= \frac{1-(\operatorname{ch}^2 a + \operatorname{ch}^2 b + \operatorname{ch}^2 c) + 2\operatorname{ch} a \operatorname{ch} b \operatorname{ch} c}{\operatorname{sh}^2 a \operatorname{sh}^2 b \operatorname{sh}^2 c}.$$

By symmetry of the RHS in a, b, c, we have

$$\left(\frac{\sin A}{\operatorname{sh} a}\right)^2 = \left(\frac{\sin B}{\operatorname{sh} b}\right)^2 = \left(\frac{\sin C}{\operatorname{sh} c}\right)^2$$

Since $A+B+C < \pi$, $(A,B,C>0)$

we have $\sin A, \sin B, \sin C > 0$.

Hence $$\frac{\sin A}{\operatorname{sh} a} = \frac{\sin B}{\operatorname{sh} b} = \frac{\sin C}{\operatorname{sh} c}.$$