Case (A)

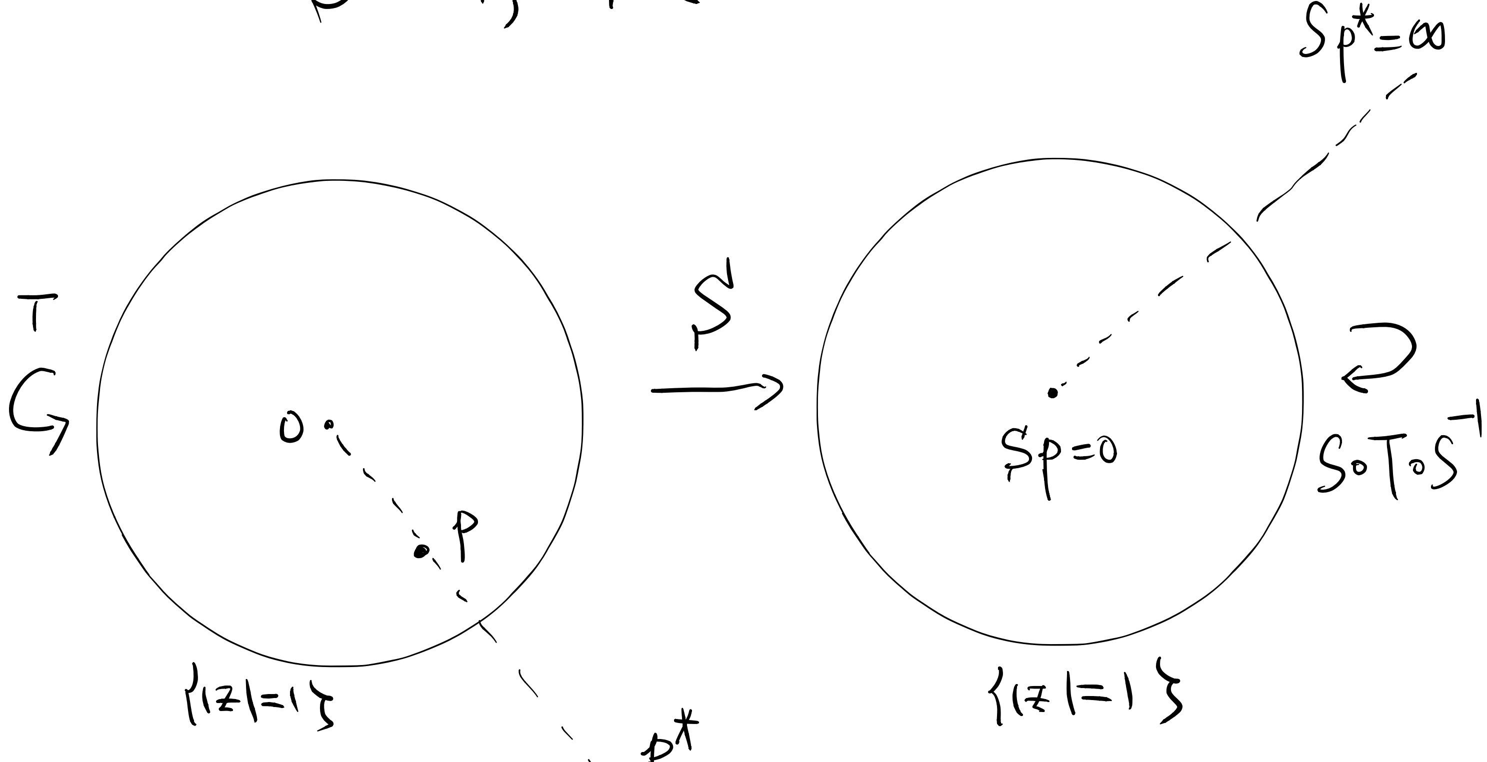
Let $p \in \mathbb{D}$ and $p^* \in \mathbb{C} \setminus \mathbb{D}$ be the fixed points of T (If $p = 0$, then $p^* = \infty$)

Let $S \in \mathbb{H}$ be a transformation in the hyperbolic group such that

$$Sp = 0$$

and hence $$Sp^* = \infty.$$

Since $S \in \mathbb{H}$, $S(\{|z|=1\}) = \{|z|=1\}$.

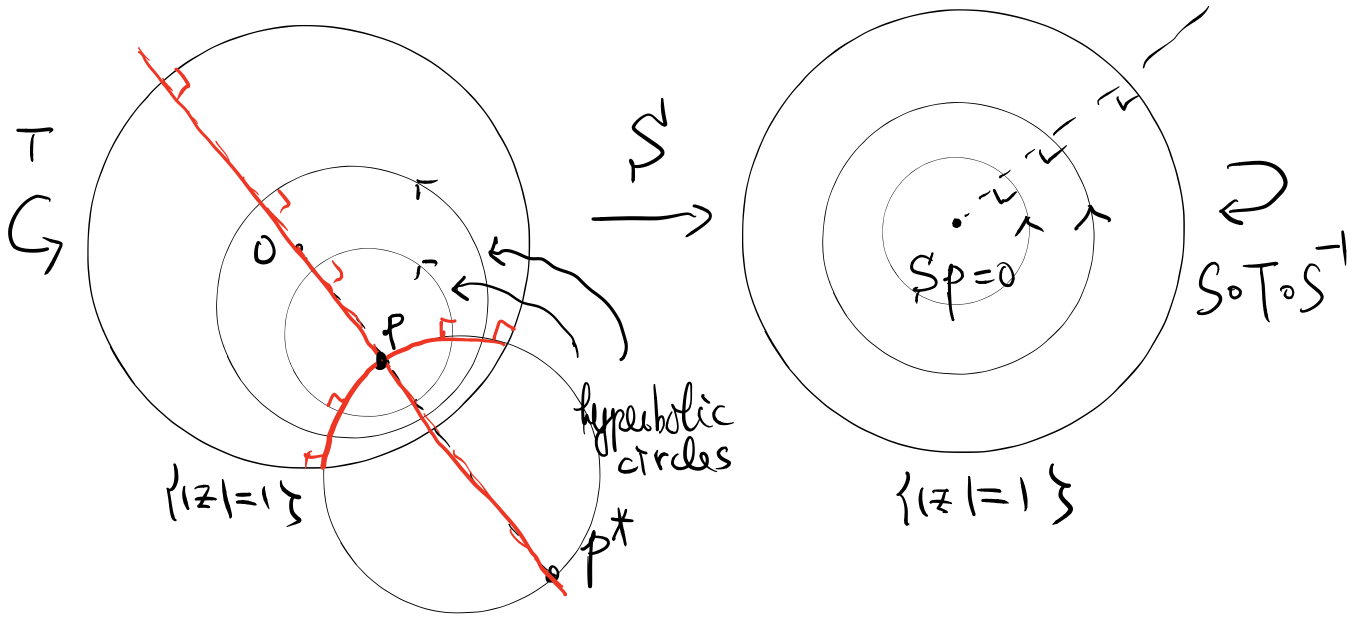


Then $S \circ T \circ S^{-1} \in \mathbb{H}$ and has fixed points at 0 and ∞, and fixing the unit circle.

Hence $S \circ T \circ S^{-1}$ is a rotation and T is a elliptic transformation. Moreover, we have the following conclusions:

(1) the unit circle is a Steiner circle of the 2nd kind wrt p and p^*.

(2) Other Steiner circles of 2nd kind (Apollonius) are hyperbolic circles.



(3) The transformation T "rotates" points around these circles of Apollonius, creating a circulating motion about the fixed point p.

(T is called a hyperbolic rotation)

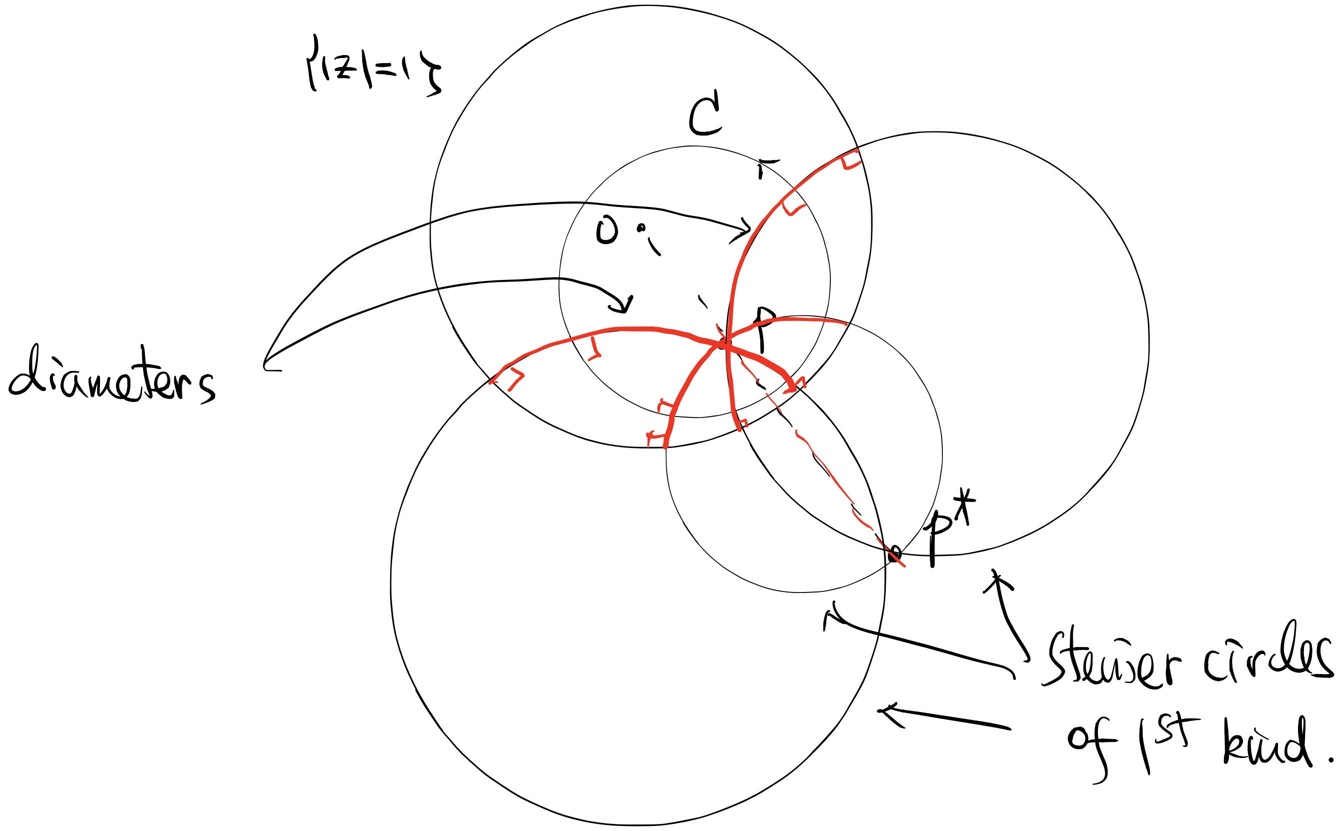
(4) The Steiner circles of 1^{st} kind wrt p and p^* are hyperbolic straight lines (passing thro. p)

Remark:

Suppose that C is a hyperbolic circle. Then the family of all circles perpendicular to C and perpendicular to the unit circle $\{|z|=1\}$ is a family of Steiner circles of the 1^{st} kind wrt some points p and p^* $(|p|<1, |p^*|>1)$.

(Ex!)

The point p is called the center of C, and the Steiner circles of the 1^{st} kind are called the diameters of C.



Case (B)

Let p and q be the 2 fixed points of T on $\{|z|=1\}$ $\Rightarrow$ T is hyperbolic

($\because$ elliptic with fixed p and q will move $\{|z|=1\}$)

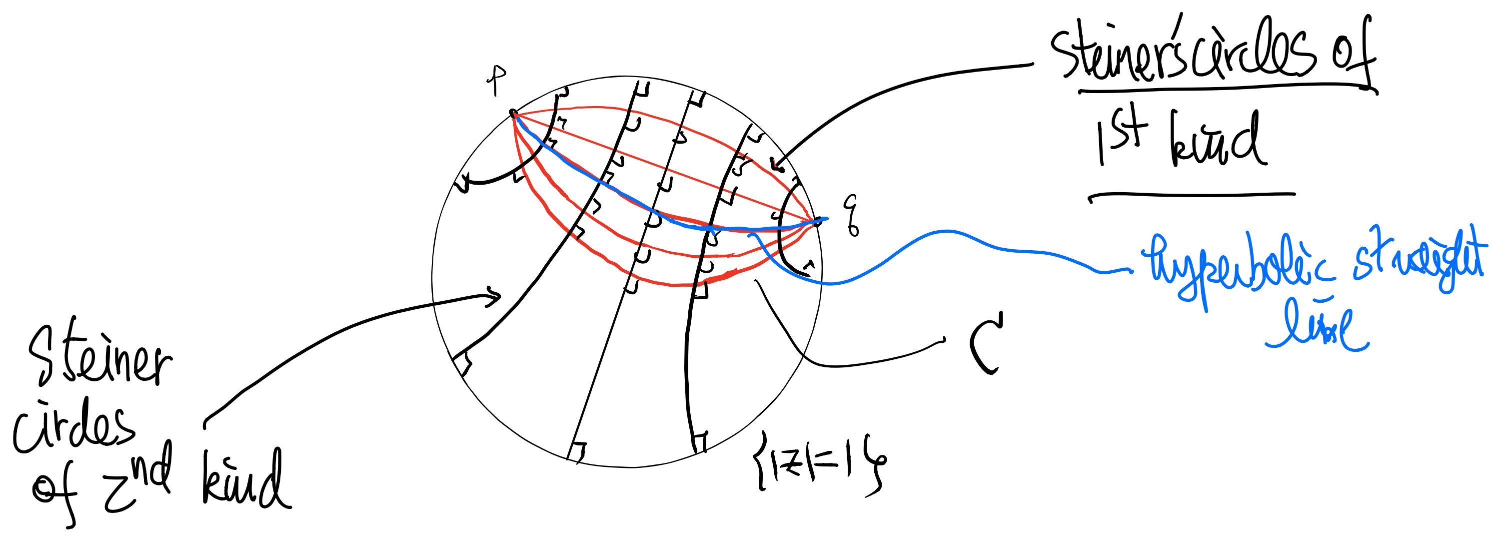
(1) Then the unit circle is a Steiner Circle of 1^{st} kind wrt p and q.

(2) Other Steiner circles of 1^{st} kind consist of

(a) a hyperbolic straight lines determined by p and q

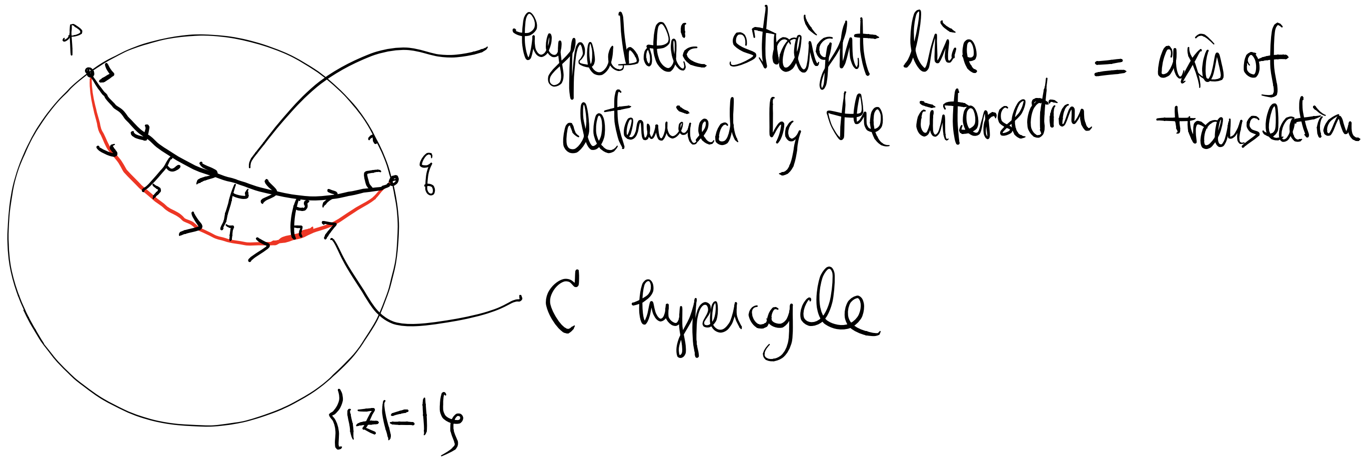
(b) hypercycles.

(3) Steiner circles of 2^{nd} kind are mutually hyperparallel hyperbolic straight lines



Remark: Conversely, if we have a hypercycle C.
Then C intersects the unit circle in 2 points, say p & q $\Rightarrow$ C is a Steiner circle of the 1^{st} kind wrt p & q.

(unit circle & the hyperbolic straight line determined by p & q are also Steiner circles of the 1^{st} kind)



The "hyperbolic distance" (see next section) of the perpendicular from C to the hyperbolic straight line is constant (i.e. independent of the starting point on C) since these perpendiculars are congruent. (Ex!)

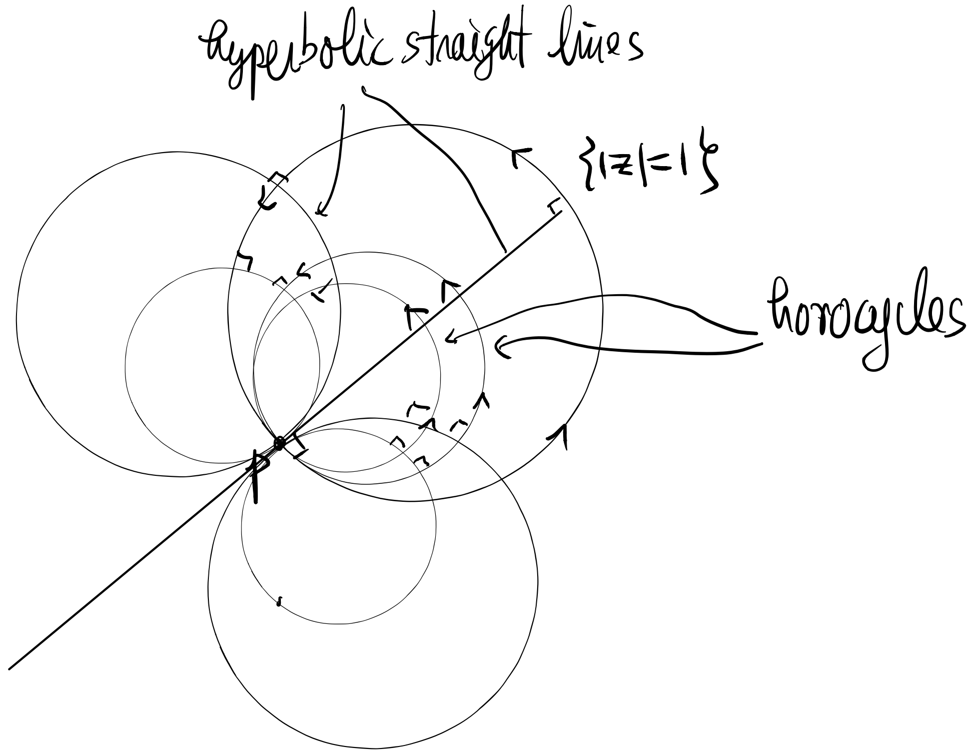
- Therefore, hypercycles are called equidistant curves.

(Ex: Compare with Euclidean geometry!)

Note that T moves points along the equidistant curves from to p to q (or q to p), and hence called a hyperbolic translation; the unique hyperbolic straight line determined by p & q is called the axis of translation.

Case (C)

T must be a parabolic transformation



(1) the hyperbolic straight lines passing thro. p are degenerated Steiner circles perpendicular to $\{|z|=1\}$

(2) The family of degenerate Steiner circles perpendicular to the hyperbolic straight lines from p are horocycles.

(degenerate Steiner circles are those circles in the generalized Cartesian coordination system corresponding to a parabolic Möbius transformation.)

(3) The hyperbolic straight lines thro. p are the diameters of the horocycles.

(4) The transformation T is called a parallel displacement.

Summary:

(A) hyperbolic circle is a curve traced out by a point subjected to elliptic transformation

$\Leftrightarrow$ Steiner circle of 2^{nd} kind wrt the fixed points of an elliptic transformation (fixing $\{|z|=1\}$)

(B) hypercycle is a curve traced out by a point subjected to hyperbolic transformation

$\Leftrightarrow$ Steiner circle of 1^{st} kind wrt the fixed points of a hyperbolic transformation (fixing $\{|z|=1\}$)

(c) horocycle is a curve traced out by a point subjected to parabolic transformation.

$\longleftrightarrow$ degenerate Steiner circle of a parabolic transformation (fixing $\{|z|=1\}$) perpendicular to hyperbolic straight lines thro. the fixed (ideal) point.

Ch9 Hyperbolic length

Def: A (parametric) curve

$$\gamma = z(t) = x(t) + i\,y(t), \quad t \in [a,b]$$

is called smooth if $x(t)$ and $y(t)$ are differentiable.

- $z(a)$ and $z(b)$ are called end points of the curve γ.

Def: In the hyperbolic plane, the length of a smooth curve γ with parametrization $z(t) = x(t) + i\,y(t)$, $a \le t \le b$, is given by

$$\boxed{\ell(\gamma) = 2\int_a^b \frac{|z'(t)|}{1-|z(t)|^2}\,dt}$$

where $z'(t) = x'(t) + i\,y'(t)$

Def: Let z_1 and z_2 be two points in the hyperbolic plane. The distance from z_1 to z_2 is defined by

$$d(z_1, z_2) = \ell\left(\begin{array}{c}\text{hyperbolic straight line segment} \\ \text{between } z_1 \text{ \& } z_2\end{array}\right)$$

Remarks: (i) $\ell(\gamma) \geq 0$, $\forall \gamma$.

(ii) $|z'(t)|dt = \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$

is just the usual Euclidean integrand for arc-length.

Thm: Let T be a transformation of hyperbolic group and γ be a smooth curve. Then

$$\ell(T(\gamma)) = \ell(\gamma).$$

i.e. "length" is invariant

$\Rightarrow$ "distance" is invariant.

Pf: Let $w = Tz = e^{i\theta} \dfrac{z - z_0}{1 - \bar{z}_0 z}$ $\left(|z_0| < 1, \theta \in \mathbb{R} \right)$

and $\gamma : z(t)$

Then $T(\gamma)$: $w(t) = Tz(t)$

$$= e^{i\theta} \frac{z(t) - z_0}{1 - \bar{z}_0 z(t)}.$$

And $w'(t) = e^{i\theta} \dfrac{1 - |z_0|^2}{\left[1 - \bar{z}_0 z(t)\right]^2} z'(t)$ (Ex!)

[Note: $T'(z) = e^{i\theta} \dfrac{1 - |z_0|^2}{(1 - \bar{z}_0 z)^2}$]

$$\Rightarrow \frac{|w'(t)|}{1 - |w(t)|^2} = \frac{1}{1 - \left| \dfrac{z(t) - z_0}{1 - \bar{z}_0 z(t)} \right|^2} \cdot \frac{1 - |z_0|^2}{\left| 1 - \bar{z}_0 z(t) \right|^2} |z'(t)|$$

$$= \frac{1 - |z_0|^2}{|1 - \bar{z}_0 z(t)|^2 - |z(t) - z_0|^2} |z'(t)|$$

$$= \frac{1 - |z_0|^2}{\left[\begin{array}{l} 1 - \bar{z}_0 z(t) - z_0 \overline{z(t)} + |z_0|^2 |z(t)|^2 \\ - |z(t)|^2 + \bar{z}_0 z(t) + z_0 \overline{z(t)} - |z_0|^2 \end{array}\right]} |z'(t)|$$

$$= \frac{1-|z_0|^2}{1-|z_0|^2-|z(t)|^2+|z_0|^2|z(t)|^2}\,|z'(t)|$$

$$= \frac{1-|z_0|^2}{(1-|z_0|^2)(1-|z(t)|^2)}\,|z'(t)|$$

$$= \frac{|z'(t)|}{1-|z(t)|^2}$$

Hence $$\ell(T\gamma) = 2\int_a^b \frac{|w'(t)|}{1-|w(t)|^2}\,dt$$

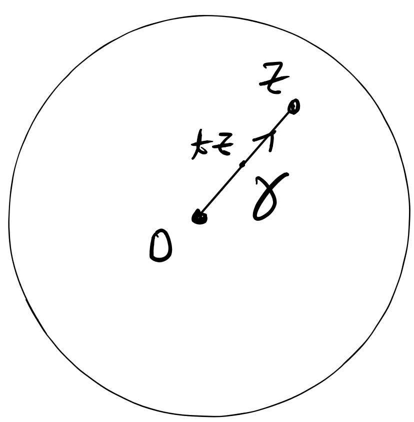
$$= 2\int_a^b \frac{|z'(t)|}{1-|z(t)|^2}\,dt = \ell(\gamma).$$

#

<u>Distance famula</u>

<u>Case 1</u> $\boxed{d(0,z) = \ln\frac{1+|z|}{1-|z|}} \boxed{= \log\frac{1+|z|}{1-|z|}}$

<u>Pf</u>: The Euclidean straight line segment is the hyperbolic straight line segment in this case. Hence $d(0,z) = \ell(\gamma)$



where $\gamma: z(t) = tz, \quad 0 \le t \le 1.$

$$\Rightarrow d(0,z) = 2\int_0^1 \frac{|z'(t)|}{1-|z(t)|^2}\,dt = 2\int_0^1 \frac{|z|}{1-|tz|^2}\,dt$$

$$= 2\int_0^{|z|} \frac{dr}{1-r^2}, \quad \text{letting } r = t|z|$$

$$= \ln \frac{1+|z|}{1-|z|}. \quad \#$$

<u>Case 2</u> (<u>general case</u>)

$$\boxed{d(z_1,z_2) = \ln\left(\frac{1+\left|\frac{z_2-z_1}{1-\bar{z}_1 z_2}\right|}{1-\left|\frac{z_2-z_1}{1-\bar{z}_1 z_2}\right|}\right)}$$

<u>Pf</u>: Let $Tz = e^{i\theta}\,\frac{z-z_1}{1-\bar{z}_1 z}$ (any θ)

Then $Tz_1 = 0$

By invariance of ℓ, we have

$$d(z_1,z_2) = d(Tz_1, Tz_2)$$

$$= d(0, Tz_2)$$

$$= \ln \frac{1+|Tz_2|}{1-|Tz_2|} \quad (\text{by case } 1)$$

$$= \ln \frac{1+\left|\frac{z_2-z_1}{1-\bar{z}_1 z_2}\right|}{1-\left|\frac{z_2-z_1}{1-\bar{z}_1 z_2}\right|}$$

\#

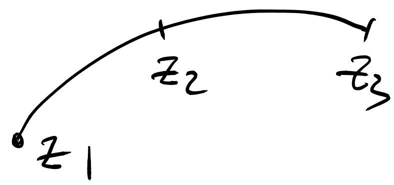
Fundamental Properties of Distance

Thm: Let z_1, z_2, z_3 be points in the hyperbolic plane.

Then (1) $d(z_1, z_2) \geq 0$

(2) $d(z_1, z_2) = d(z_2, z_1)$

(3) If z_1, z_2 and z_3 are colinear (in the order)



then

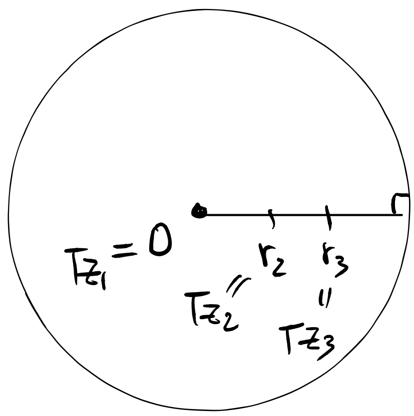
$$d(z_1, z_3) = d(z_1, z_2) + d(z_2, z_3).$$

(colinear : z_1, z_2, z_3 on the same hyperbolic straight line)

Pf: (1) & (2) are clear.

Pf of (3): Let T be a transformation of hyperbolic geometry taking z_1 to 0 and the hyperbolic straight line passing thro. z_1, z_2 & z_3 to the (positive) x-axis. Then $Tz_1 = 0$, $Tz_2 = r_2$ and $Tz_3 = r_3 \geq r_2$

(with $r_2, r_3 \in [0,1)$)



Then

$$d(z_1, z_3) = d(0, r_3)$$
$$= 2\int_0^{r_3} \frac{dr}{1-r^2}$$
$$= 2\int_0^{r_2} \frac{dr}{1-r^2} + 2\int_{r_2}^{r_3} \frac{dr}{1-r^2}$$
$$= d(0, r_2) + 2\int_{r_2}^{r_3} \frac{dr}{1-r^2}$$

Note $d(z_2, z_3) = d(r_2, r_3) = \ln \dfrac{1 + \left|\dfrac{r_3 - r_2}{1 - \bar{r}_2 r_3}\right|}{1 - \left|\dfrac{r_3 - r_2}{1 - \bar{r}_2 r_3}\right|}$

$$= \ln \frac{1+\left|\frac{r_3-r_2}{1-r_2r_3}\right|}{1-\left|\frac{r_3-r_2}{1-r_2r_3}\right|} = \ln \frac{1-r_2r_3+r_3-r_2}{1-r_2r_3-r_3+r_2}$$

(since $0 \le r_2 \le r_3 < 1$)

$$= \ln \frac{(1+r_3)(1-r_2)}{(1-r_3)(1+r_2)} = \ln \frac{1+r_3}{1-r_3} - \ln \frac{1+r_2}{1-r_2}$$

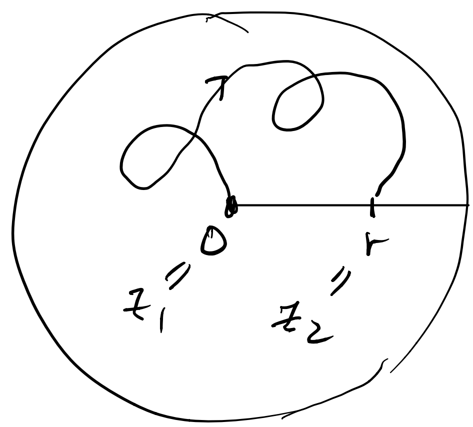
$$= 2\int_{r_2}^{r_3} \frac{dr}{1-r^2} .$$

$\therefore \ d(z_1, z_3) = d(z_1, z_2) + d(z_2, z_3)$. ※

Thm Let z_1 & z_2 be points in the hyperbolic plane. Then the shortest curve (in hyperbolic length) connecting z_1 & z_2 is the hyperbolic straight line segment joining z_1 & z_2.

Pf of the Thm

We may assume $z_1 = 0$ and $z_2 = r \in (0,1)$ (by a transformation)



Let γ be a (smooth) curve joining 0 and r with parametrization

$$z(t) = x(t) + i\,y(t), \quad a \le t \le b$$

with
$$\begin{cases} 0 = z_1 = z(a) = x(a) + i\,y(a) \\ r = z_2 = z(b) = x(b) + i\,y(b) \end{cases}$$

$$\Longrightarrow \begin{cases} x(a) = y(a) = y(b) = 0 \\ x(b) = r \end{cases}$$

Then
$$\ell(\gamma) = 2\int_a^b \frac{|z'(t)|}{1 - |z(t)|^2}\,dt$$

$$= 2\int_a^b \frac{\sqrt{(x'(t))^2 + (y'(t))^2}}{1 - (x(t))^2 - (y(t))^2}\,dt$$

$$\geq 2\int_a^b \frac{\sqrt{(x'(t))^2}}{1-x(t)^2}\,dt$$

$$= 2\int_a^b \frac{|x'(t)|}{1-x(t)^2}\,dt$$

$$\geq 2\int_a^b \frac{x'(t)}{1-x(t)^2}\,dt$$

$$= 2\int_{x(a)}^{x(b)} \frac{ds}{1-s^2}$$ where $s = x(t)$

$\Rightarrow ds = x'(t)\,dt$

$t = a \to s = x(a)$
$t = b \to s = x(b)$

$$= 2\int_0^r \frac{ds}{1-s^2}$$

$$= \ln\frac{1+r}{1-r}$$

$$= d(0,r) = d(z_1, z_2)$$

$\therefore\ l(\gamma) \geq d(z_1, z_2) = l\left(\begin{array}{c}\text{hyperbolic straight line}\\ \text{segment joining } z_1 \,\&\, z_2\end{array}\right)$

#

<u>Note</u>: If fact, if $l(\gamma) = d(0,r)$

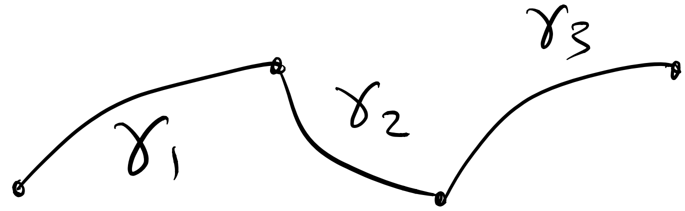
then $y' \equiv 0$ and $y \equiv 0$

and $\quad x' = |x'| > 0$

$\Rightarrow$ after transformation, γ = x-axis between 0 and r (in the increasing direction)

$\Rightarrow$ γ = hyperbolic straight line segment joining z_1 & z_2

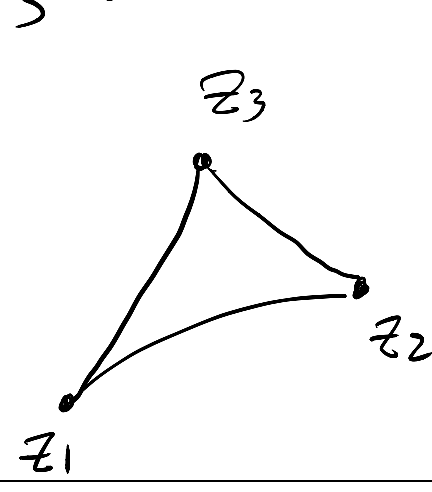
<u>Remark</u>: The Thm (& the Note) is actually true for piecewise smooth curves:



<u>Corollary</u> (<u>Triangle Inequality</u>)

For any 3 points z_1, z_2 & z_3 in the hyperbolic plane,

$$d(z_1, z_3) \leq d(z_1, z_2) + d(z_2, z_3)$$



(Pf: Ex!)