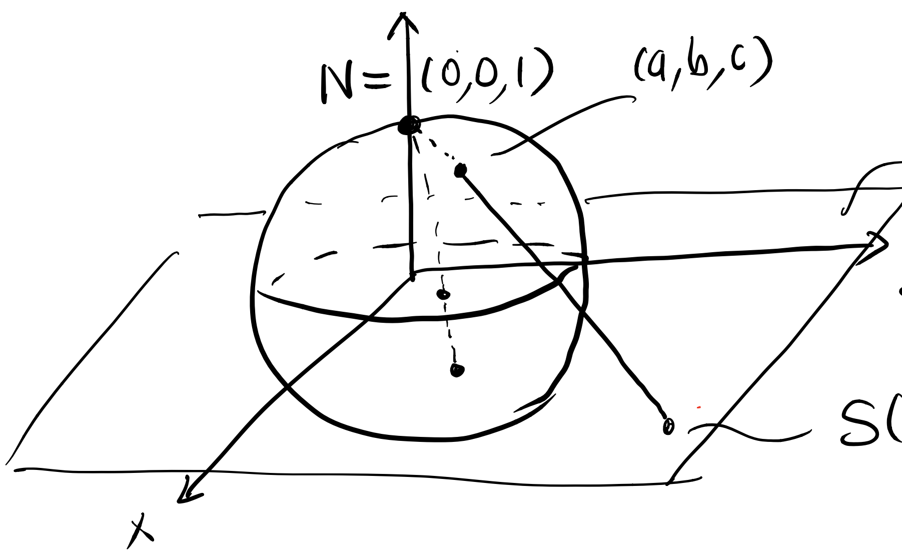
Point at ∞: One can add an ∞ to $\mathbb{C}$ in correspondence with $N=(0,0,1)\in S^2$



$\sim x+iy = \dfrac{a+ib}{1-c}$

as $c\to 1$, $\underset{\in S^2}{(a,b,c)} \to (0,0,1)$

and $S(a,b,c)=x+iy = \dfrac{a+ib}{1-c} \to \infty$

in the sense that $\left|\dfrac{a+ib}{1-c}\right| \to +\infty$.

Note: We only have 1 complex ∞. Therefore, it is different from real number situation.

Notation: $\mathbb{C}\cup\{\infty\}$ is usually denoted by $\hat{\mathbb{C}}$ (or $\mathbb{C}^+$) and called the extended complex plane.

Note:

$$|S(a,b,c)| = \left|\frac{a+ib}{1-c}\right| = \frac{\sqrt{a^2+b^2}}{1-c}$$

but $(a,b,c) \in S^2$

$$a^2+b^2+c^2 = 1$$

$$\Rightarrow \quad a^2+b^2 = 1-c^2$$

$$\sqrt{a^2+b^2} = \sqrt{1-c^2}$$

$$\therefore \quad |S(a,b,c)| = \frac{\sqrt{a^2+b^2}}{1-c} = \frac{\sqrt{(1+c)(1-c)}}{1-c} = \sqrt{\frac{1+c}{1-c}}$$

which implies $(a,b,c) \to (0,0,1)$ if and only if $|S(a,b,c)| \to +\infty$.

eg: Inversion $z \mapsto \frac{1}{z}$ can be considered as transformation on $\hat{\mathbb{C}}$ defined by

$$\begin{cases} z \mapsto \frac{1}{z}, & \forall z \in \mathbb{C} \setminus \{0\} \\ 0 \mapsto \infty \\ \infty \mapsto 0 \end{cases}$$

Def: Lifts (of transformations)

(1) Let $S: D \to R$ be surjective continuous map. We say that S is a covering transformation from D to R, or that D covers R via S.

(2) Let $f: R \to R$ be a transformation. A transformation $g: D \to D$ is a lift of f if $\forall z \in D$, we have $\boxed{S(g(z)) = f(S(z))}$

eg: (i) Stereographic projection $S: S^2 \setminus \{N\} \to \mathbb{C}$ is a covering transformation.

(ii) Extending stereographic projection by

$$S: \quad S^2 \longrightarrow \hat{\mathbb{C}}$$

$$\begin{cases} (a,b,c) \in S^2 \setminus \{N\} \longmapsto S(a,b,c) \in \mathbb{C} \\ N = (0,0,1) \longmapsto \infty \in \hat{\mathbb{C}} \end{cases}$$

is also a covering transformation.

Remark: (2) in the Definition can be presented as the following figure (commutative diagram)

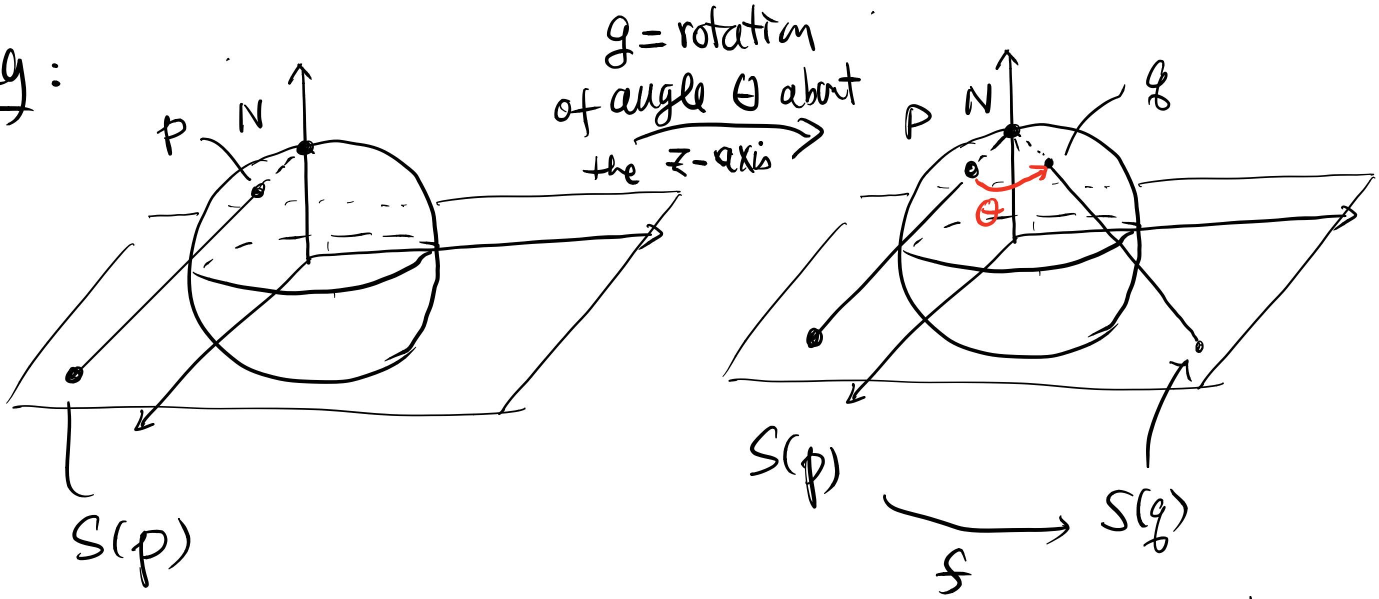
$$\begin{tikzcd} z \in D \arrow[r, dashed, "g"] \arrow[d, "S"'] & D \ni g(z) \arrow[d, "S"] \\ S(z) \in R \arrow[r, "f"] & R \end{tikzcd}$$

$$f(S(z)) = S(g(z)) \quad \forall z \in D$$

Note: S^{-1} may not exist, since it is assumed to be surjective continuous, not necessary injective.

eg: $\mathbb{C} \to \mathbb{C}$, $z \mapsto z^2$ is a covering transformation which is not invertible.

(covering transformation may not be a (geometric) transformation) eg:



is infact the rotation of the same angle (about o) in the complex plane.

Therefore

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
S^2 \arrow[r, "g = \text{rotation of}", "\text{angle } \theta \text{ about the } z\text{-axis}"'] \arrow[d, "\substack{\text{Stereographic} \\ \text{projection}}\ S"'] \arrow[dr, phantom, "\circlearrowright"] & S^2 \arrow[d, "S"] \\
\hat{\mathbb{C}} \arrow[r, "f = \text{rotation of}", "\text{angle } \theta \text{ about the origin.}"'] & \hat{\mathbb{C}}
\end{tikzcd}
$$

$(f(\infty) = \infty)$

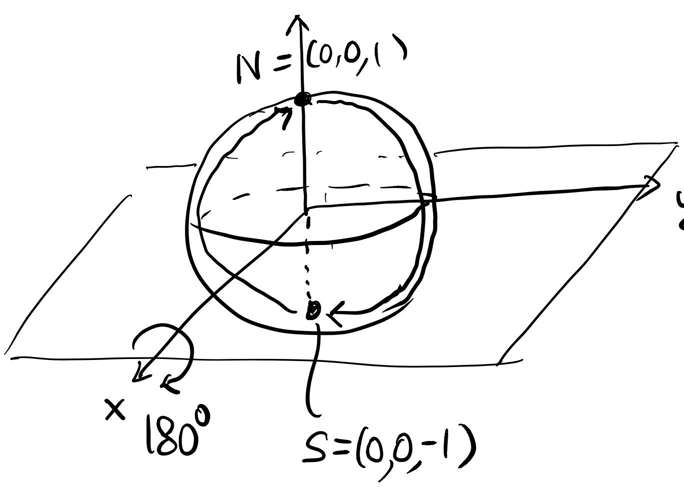
$\therefore$ g is a <u>lift</u> of f. ※

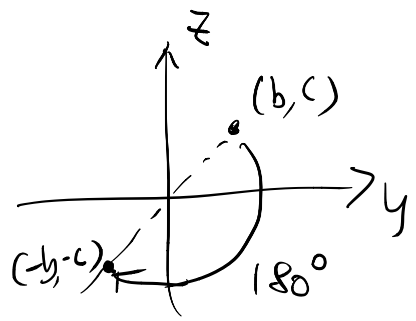
eg: • Inversion $T: \mathbb{C}\setminus\{0\} \to \mathbb{C}\setminus\{0\}$

$$z \mapsto w = \frac{1}{z}$$

• Stereographic projection $S: \mathbb{S}^2\setminus\{N,S\} \to \mathbb{C}\setminus\{0\}$ is a covering transformation. ($S=(0,0,-1)$ the South pole)

Then T lifts to a rotation of $\underline{180^\circ}$ on the $\mathbb{S}^2$ about the $\underline{x\text{-axis}}$ via the Stereographic projection S.





$$\begin{cases} \mathbb{S}^2\setminus\{N,S\} \to \mathbb{S}^2\setminus\{N,S\} \\ N=(0,0,1) \mapsto S=(0,0,-1) \\ S=(0,0,-1) \mapsto N=(0,0,1) \end{cases}$$

Pf :

south pole $(0,0,-1)$

$$
\begin{tikzcd}
(a,b,c) \in S\setminus\{N,S\} \arrow[r, "\text{rotation of } 180^\circ", "\text{wrt } X\text{-axis}"'] \arrow[d, "s"'] \arrow[dd, bend right] & S\setminus\{N,S\} \arrow[d, "s"] \\
\mathbb{C}\setminus\{0\} \arrow[r, "T"] & \mathbb{C}\setminus\{0\} \\
z \arrow[r, mapsto] & Tz = \frac{1}{z}
\end{tikzcd}
$$

Let $z = \dfrac{a+ib}{1-c}$ be the image of the point $(a,b,c) \in S^2\setminus\{N,S\}$ under the stereographic projection.

$$Tz = \frac{1}{z} = \frac{1}{\frac{a+ib}{1-c}} = \frac{1-c}{a+ib}$$

$$= \frac{(1-c)(a-ib)}{a^2+b^2}$$

$$= \frac{(1-c)(a-ib)}{1-c^2} = \frac{a-ib}{1+c}$$

$$= \frac{a+i(-b)}{1-(-c)} = S(a,-b,-c)$$

i.e. $T\big(S(a,b,c)\big) = S(a,-b,-c)$

Let g = rotation of 180° with respect to the x-axis,

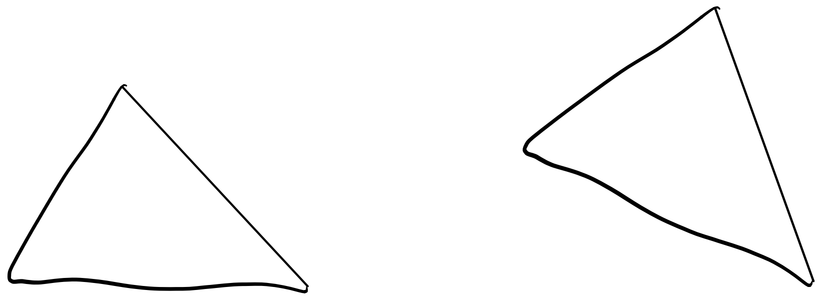
then $g(a,b,c) = (a,-b,-c)$.

Hence $T(S(a,b,c)) = S(g(a,b,c))$

$\therefore$ g is lift of T. ※

Ch4 The Erlanger Program (Klein)

Congruence and transformation



In history, 2 figures are congruent when one can be moved so as to coincide with the other.

move $\longleftrightarrow$ transformation

Klein's idea: "congruence" determines "geometry"

$\updownarrow$

(need "transformation" to define)

Classical congruence relation of Euclidean geometry satisfy:

(a) (reflexivity) $A \cong A$ for any figure A

(b) (symmetry) If $A \cong B$, then $B \cong A$

(c) (transitivity) If $A \cong B$ & $B \cong C$, then $A \cong C$.

("$\cong$" congruent)

Remark: A relation with properties (a), (b) & (c) is called an equivalence relation.

Definition of Geometry (in the sense of Klein)

The properties of the classical congruence relation can be expressed in terms of properties of congruence transformations:

Set of transformations $\{f\}$ such that

$$A \cong B \Leftrightarrow A = f(B) = \{f(b) : b \in B\} \quad \text{for some } f$$

Then (a) $f(z) = z$ (identity transformation) is a congruence transformation.

(b) If $f(z)$ is a congruence transformation, then f is invertible and $f^{-1}(z)$ is also a congruence transformation.

(c) If $f(z)$ and $g(z)$ are congruence transformations, then so is the composition $(f \circ g)(z) = f(g(z))$.

Def: Let S be a non-empty set. A transformation group is a collection G of transformations $f: S \to S$ such that

(a) G contains the identity Id_S. $(Id_S \in G)$

(b) the transformations in G are invertible and their inverses are in G.

$(\forall f \in G,\ f^{-1}$ exists and $f^{-1} \in G)$

(c) G is closed under composition.

$(\forall f, g \in G,\ f \circ g \in G)$

Def: • A geometry is a pair (S, G) consisting of a nonempty set S and a transformation group G (on S)

- The set S is the underlying space of the geometry
- The set G is the transformation group of the geometry

Def:
- A *figure* is any subset A of the underlying space S of the geometry (S, G).
- 2 figures A & B are *congruent* if there is a transfamation $T \in G$ such that $T(A) = B$, where $T(A) = \{Tz : z \in A\}$

---

eg (1) Euclidean Geometry (without reflections) (plane)

Underlying space S = complex plane $\mathbb{C}$ ($\neq \emptyset$)

Transfamation group G

= set E of transfamations of the fam

$$Tz = e^{i\theta} z + b \quad (\theta \in \mathbb{R}, b \in \mathbb{C})$$

(if we want to include reflections, then we also need to include transfamations of the fam $e^{i\theta} \bar{z} + b$.)

"T" is called a *rigid motion*

= composition of rotation & translation.

The pair $(\mathbb{C}, E)$ <u>models</u> Euclidean geometry (without reflections)

<u>Check</u>: E is a transformation group

(a) $Id_{\mathbb{C}}: z \mapsto z \in E$ (with $\theta = 0$ & $b = 0$)

(b) If $Tz = e^{i\theta}z + b$, then

$$T^{-1}z = e^{-i\theta}(z - b)$$
$$= e^{i(-\theta)}z + (-e^{-i\theta}b) \in E$$

(c) If $T_1 z = e^{i\theta_1}z + b_1$

$T_2 z = e^{i\theta_2}z + b_2$

then
$$(T_1 \circ T_2)(z) = e^{i\theta_1}(T_2 z) + b_1$$
$$= e^{i\theta_1}(e^{i\theta_2}z + b_2) + b_1$$
$$= e^{i(\theta_1+\theta_2)}z + (e^{i\theta_1}b_2 + b_1) \in E$$

#

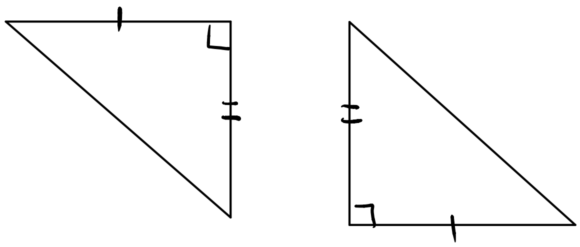
eg(2) Translational Geometry

Underlying space $= \mathbb{C}$

Transformation group $\mathbb{T} = \{ Tz = z+b : b \in \mathbb{C} \}$

Then $(\mathbb{C}, \mathbb{T})$ is a geometry (Ex: check!)

Note: in translational geometry



are not congruent in $(\mathbb{C}, \mathbb{T})$.

eg(3) The Trivial Geometry

$(S, \{Id_S\})$ i.e. G consists of identity element only.

Note: 2 "different" figures are never congruent in this geometry.

Invariant

Def: Let (S, G) be a geometry.

Let D be a set of figures from (S, G)

(i.e. elements of D are subsets of S)

- The set D is invariant (in the geometry (S, G)) if $\forall B \in D$ & $T \in G$, then $T(B) \in D$.

- A function f defined on D (← invariant) is called invariant (in the geometry (S, G)) if

$$f(T(B)) = f(B), \quad \forall B \in D \text{ \& } T \in G.$$

egs:

(1) Triangles

$D = \{\text{triangles in } \mathbb{C}\}$ is invariant in the Euclidean Geometry.

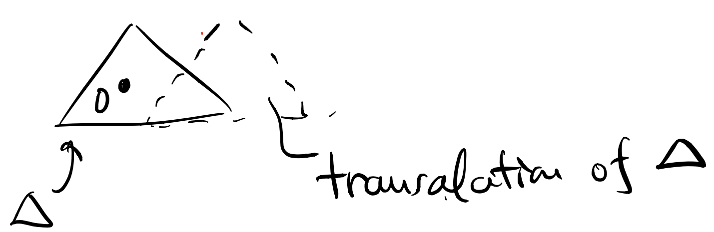
(2) Area, Perimeter of triangles are invariant functions in Euclidean geometry: $f = \text{Area}: D \to \mathbb{R}$ and $f = \text{perimeter}: D \to \mathbb{R}$ are invariant functions.

(3) $D = \{\text{triangles in } \mathbb{C}\}$

d = sum of distance of vertexes to the origin

d is <u>not</u> <u>invariant</u> in the Euclidean geometry

$$d(\Delta) \neq d(\text{translation of } \Delta)$$



> **<u>Erlanger Program</u> (Klein)**
>
> The subject matter of a geometry is its invariant sets and the invariant functions on those sets.

<u>eg</u>: We study triangles and its area, perimeter, etc in the Euclidean geometry.

Geometric Proof

Let (S, G) = a geometry

If (i) F is a figure in S such that statement "W" is true.

(ii) all measurements and other quantities mentioned in the statement "W" are invariant.

Then $\forall\, T \in G$, "W" is also true for $T(F)$.

<u>Application</u>: To prove "W" is true for F, we find a $T \in G$ such that "W" is easy to prove for $T(F)$.