Ch 7 The 1^{st} & 2^{nd} variation formula

Let
- M = complete Riem. mfd
- $\gamma(t,u) : [a,b] \times [-\varepsilon, \varepsilon] \to M$ a C^∞ map
- $\{\gamma_u(t)\}$ corresponding 1-parameter family of curves with base curve γ_0 equal to a given curve $\gamma(t)$ parametrized by arc-length, i.e. $|\gamma'(t)| = 1$.
- U = transversal vector field of $\{\gamma_u\}$.
- T = tangent vector field along $\{\gamma_u\}$.

Then the length of $\gamma_u(t)$ is

$$L(u) = \int_a^b |\gamma_u'(t)|\,dt = \int_a^b |T|\,dt$$

$$\Rightarrow \quad \frac{dL}{du}(u) = \int_a^b \frac{d}{du}|T|\,dt = \int_a^b U\sqrt{\langle T,T\rangle}\,dt$$

$$= \int_a^b \frac{\langle T, D_U T\rangle}{|T|}\,dt$$

$$= \int_a^b \frac{1}{|T|}\langle T, D_T U\rangle\,dt \quad \left([T,U]=0\right)$$

—— $(*)_1$

$$\Rightarrow \quad \frac{dL}{du}(0) = \int_a^b \langle \gamma'(t), D_{\gamma'(t)} U \rangle \, dt$$

$$= \int_a^b \left[\frac{d}{dt} \langle \gamma'(t), U \rangle - \langle D_{\gamma'(t)} \gamma'(t), U \rangle \right] dt$$

where $U(t) = U(t,0)$ is the transversal vector field along γ.

$$\boxed{\frac{dL}{du}(0) = \langle \gamma'(t), U(t) \rangle \Big|_a^b - \int_a^b \langle D_{\gamma'(t)} \gamma'(t), U \rangle \, dt}$$

which is the 1^{st} variation formula for arc-length.

Lemma 1: A curve $\gamma : [a,b] \to M$ is a geodesic

$\Leftrightarrow$ it is a critical point of the arc-length functional with respect to (all) normal variations $\{\gamma_u\}$

(i.e. $\forall u$, $\gamma_u(a) = \gamma(a)$ & $\gamma_u(b) = \gamma(b)$)

Pf: For normal variations, $U(a) = U(b) = 0$.

$$\therefore \quad \frac{dL}{du}(0) = - \int_a^b \langle D_{\gamma'} \gamma', U \rangle \, dt$$

$\forall\, U$ with $U(a) = U(b) = 0$.

$$\therefore \quad 0 = \frac{dL}{du}(0) \quad \Leftrightarrow \quad D_{\gamma'} \gamma' = 0 \quad (\text{Ex!})$$

$\forall\, U$ with $U(a) = U(b) = 0$

$\divideontimes$

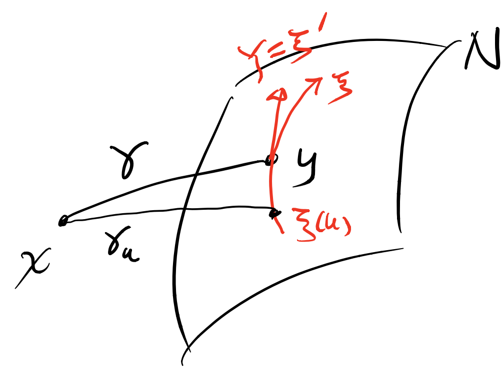
Lemma 2 Let
- N = closed submanifold of M
- $x \notin N$
- $y \in N$ such that

$$d(x,y) = d(x,N) \overset{\text{def}}{=} \inf_{y \in N} d(x,y)$$

- $\gamma : [a,b] \to M$ shortest geodesic joining x to y. $(\gamma(a) = x,\ \gamma(b) = y)$

Then γ is <u>normal</u> to N (i.e. $\gamma'(b) \perp T_y N$).

<u>Pf</u>: Let $Y \in T_y N$.



We need to show that $\langle \gamma'(b), Y \rangle = 0$

For this, we take a C^∞ curve $\xi \in [-\varepsilon, \varepsilon] \to N$ such that

$$\xi'(0) = Y \quad (\& \ \xi(0) = y)$$

Let $\{\gamma_u\}$ be a 1-parameter family of curves given by

$\gamma(t,u) : [a,b] \times [-\varepsilon, \varepsilon] \to M$ with

$$\begin{cases} \gamma_0(t) = \gamma(t), & \forall t \in [a,b] \\ \gamma_u(a) = x, \quad \gamma_u(b) = \xi(u), & \forall u \end{cases}$$

By assumption

$$L(0) = d(x,y) \leq d(x, \xi(u)) \leq L(u), \quad \forall u \in [-\varepsilon, \varepsilon]$$

$$\Rightarrow \quad \frac{dL}{du}(0) = 0.$$

1^{st} variation formula $\Rightarrow$

$$0 = \langle \gamma'(t), U(t) \rangle \Big|_a^b - \int_a^b \langle \cancel{D_{\gamma'}\gamma'}, U \rangle \, dt$$

$$= \langle \gamma'(b), U(b) \rangle - \langle \gamma'(a), \cancel{U(a)} \rangle$$

$$= \langle \gamma'(b), U(b) \rangle$$

By $\gamma_u(b) = \xi(u)$, $\forall u$, we have

$$U(b) = \xi'(0) = Y$$

$$\therefore \quad \langle \gamma'(b), Y \rangle = 0. \quad \text{※}$$

2nd Variation

Suppose that $\gamma : [a,b] \to M$ is a normalized geodesic.

We would like to calculate $\frac{d^2L}{du^2}(0)$ for a family $\{\gamma_u\}$.

By $(*)_1$: $$\frac{dL}{du}(u) = \int_a^b \frac{1}{|T|}\langle T, D_T U\rangle\, dt$$

$$\Rightarrow \frac{d^2L}{du^2}(u) = \int_a^b \frac{\partial}{\partial u}\left[\frac{1}{|T|}\langle T, D_T U\rangle\right] dt$$

$$= \int_a^b \left[-\frac{1}{|T|^3}\langle T, D_T U\rangle^2 + \frac{1}{|T|} U\langle T, D_T U\rangle\right] dt$$

$$= \int_a^b \left[\begin{array}{l} -\frac{1}{|T|^3}\langle T, D_T U\rangle^2 + \frac{1}{|T|}\langle D_U T, D_T U\rangle \\ \qquad + \frac{1}{|T|}\langle T, D_U D_T U\rangle \end{array}\right] dt$$

$$= \int_a^b \left[\begin{array}{l} -\frac{1}{|T|^3}\langle T, D_T U\rangle^2 + \frac{1}{|T|}|D_T U|^2 \\ + \frac{1}{|T|}\langle T, D_T D_U U + R_{TU} U\rangle \end{array}\right] dt$$

$([T, U] = 0)$

$$= \int_a^b \left\{\begin{array}{l} -\frac{1}{|T|^3}\left[T\langle T, U\rangle - \langle D_T T, U\rangle\right]^2 + \frac{1}{|T|}|D_T U|^2 \\ + \frac{1}{|T|}\left[T\langle T, D_U U\rangle - \langle D_T T, D_U U\rangle\right] \\ \qquad - \frac{1}{|T|}\langle R_{UT} U, T\rangle \end{array}\right\} dt$$

Note that at $u=0$, $\begin{cases} D_T T = D_{\gamma'}\gamma' = 0 \\ |T| = |\gamma'| = 1 \end{cases}$

$$\therefore \quad \frac{d^2L}{du^2}(0) = \int_a^b \Big[-\big[\tfrac{d}{dt}\langle \gamma', V\rangle\big]^2 + |V'|^2 + \tfrac{d}{dt}\langle \gamma', D_V V\rangle - \langle R_{V\gamma'} V, \gamma'\rangle \Big]\, dt$$

$\Rightarrow$

$$\boxed{\frac{d^2L}{du^2}(0) = \langle \gamma', D_V V\rangle\Big|_a^b + \int_a^b \Big\{ \Big(|V'|^2 - \big[\tfrac{d}{dt}\langle \gamma', V\rangle\big]^2 \Big) - \langle R_{V\gamma'} V, \gamma'\rangle \Big\}\, dt}$$

which is the 2nd variation formula (for normalized geodesic)

Let $V^\perp = V - \langle V, \gamma'\rangle \gamma'$ the normal component of V,

then the 2nd variation formula can be written as

$$\boxed{\frac{d^2L}{du^2}(0) = \langle \gamma', D_V V\rangle\Big|_a^b + \int_a^b \Big\{ |D_{\gamma'} V^\perp|^2 - \langle R_{V^\perp\gamma'} V^\perp, \gamma'\rangle \Big\}\, dt}$$

Note:
- If $\{\gamma_u\}$ is normal in the sense that

  $\gamma_u(a) = \gamma(a)\ , \quad \gamma_u(b) = \gamma(b)$

  then $\langle \gamma', D_V V\rangle(a) = \langle \gamma', D_V V\rangle(b) = 0$.

- If $\{\gamma_u\}$ is a 1-parameter of (smooth) closed curves, then $\langle \gamma', D_V V\rangle\big|_a^b = 0$.

- The interior term

$$\int_a^b \left[|D_{\gamma'} V^\perp|^2 - \langle R_{V^\perp \gamma'} V^\perp, \gamma' \rangle \right] dt$$

is related to the Jacobi Operator on $V^\perp$ (under a suitable boundary condition).

Application 1

Thm 3 Let
- M = complete simply-connected Riem. mfd. with
- $K \leq 0$ (sectional curvature)
- $O \in M$ is a fixed point.
- $\rho : M \to [0, \infty)$ (the distance function wrt O) is defined by $\rho(x) = d(x, O)$.

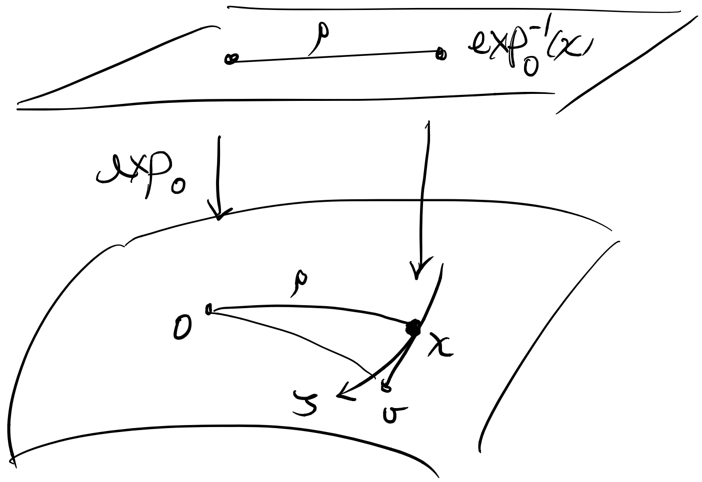
Then $\rho^2 \in C^\infty(M)$ and $D^2 \rho^2 > 0$ (strictly positive definite)

Pf: By Cartan-Hadamard Thm,

$$\rho(x) = |(\exp_O)^{-1}(x)|$$

Therefore $\rho^2(x) = |(\exp_O)^{-1}(x)|^2$ is smooth.

Now suppose $x \neq 0$ and $v \in T_x M$, $v \neq 0$. Take a curve



$$\xi = [-\varepsilon, \varepsilon] \to M$$

such that $\xi(0) = x$, $\xi'(0) = v$.

For each $u \in [-\varepsilon, \varepsilon]$, let

$$\gamma_u = [0, b] \to M \quad (\text{with } b = \rho(x),\ a = 0)$$

is the unique geodesic joining 0 to $\xi(u)$.

Note that $\gamma_0 = \gamma : [0, b] \to M$ is a normalized geodesic

(other γ_u may not be normalized.)

Also, we can choose $\xi(u)$ to be <u>a geodesic</u>. Then the end point of γ_u is $\gamma_u(b) = \xi(u)$

$\Rightarrow$ the transversal vector field $V(t, u)$ at $t = b$ is $V(b, u) = \xi'(u)$

Therefore $\quad D_V V \big|_{(b,u)} = D_{\xi'(u)} \xi'(u) = 0$.

On the other hand, $\gamma_u(0) = 0 \Rightarrow V(0,u) \equiv 0$

$$\Rightarrow D_V V \big|_{(0,u)} = 0.$$

Hence, the 2nd variation famula gives

$$\frac{d^2L}{du^2}(0) = \int_0^b \left\{ |D_{\gamma'} V^\perp|^2 - \langle R_{V^\perp \gamma'} V^\perp, \gamma' \rangle \right\} dt$$

$$\geq \int_0^b |D_{\gamma'} V^\perp|^2 \qquad (\text{since } K \leq 0)$$

Now $$D^2\rho^2(V,V) = \left\{ \zeta'(\zeta'\rho^2) - (\cancel{D_{\zeta'}\zeta'})\rho^2 \right\}\Big|_{u=0}$$

$$(\zeta = \text{geodesic})$$

$$= \zeta'(\zeta'\rho^2)\big|_{u=0}$$

$$= \zeta'(2\rho\,\zeta'\rho)\big|_{u=0}$$

$$= \left[2\rho\,\zeta'(\zeta'\rho) + 2(\zeta'\rho)^2\right]\Big|_{u=0}$$

$$= 2\rho(x)\frac{d^2}{du^2}\Big|_{u=0}\rho(\zeta(u)) + 2\left[\frac{d}{du}\Big|_{u=0}\rho(\zeta(u))\right]^2$$

Note that $\rho(\zeta(u)) = L(\gamma_u) = L(u)$

$$\Rightarrow \quad \frac{d}{du}\Big|_{u=0}\rho(\zeta(u)) = \frac{dL}{du}(0)$$

$$= \langle \gamma', V\rangle\Big|_0^b - \int_a^b \langle \cancel{D_{\gamma'}\gamma'}, V\rangle dt$$

$$= \langle \gamma'(b), U(b) \rangle$$

$$= \langle \gamma'(b), \zeta'(0) \rangle$$

$$= \langle \gamma'(b), v \rangle$$

& $$\frac{d^2}{du^2}\Big|_{u=0} \rho(\zeta(u)) = \frac{d^2 L}{du^2}(0) \geq \int_0^b |D_{\gamma'} U^{\perp}|^2 dt$$

$$\therefore \quad D^2\rho^2(v,v) \geq 2\rho(x) \int_0^b |D_{\gamma'} U^{\perp}|^2 dt + 2 \langle \gamma'(b), v \rangle^2$$

If $\langle \gamma'(b), v \rangle \neq 0$, then $D^2\rho^2(v,v) > 0$

If $\langle \gamma'(b), v \rangle = 0$, then $U(b) = v \in [\gamma'(b)]^{\perp}$

Note that $\{\gamma_u\}$ is a 1-para. family of geodesics, U is a Jacobi field along γ. Hence

$$\langle \gamma'(b), U(b) \rangle = \langle \gamma'(0), U(0) \rangle = 0$$

$\Rightarrow$ $U(t)$ is a nontrivial normal Jacobi field

$(U(b) = v \neq 0)$

$\Rightarrow$ $U^{\perp}(t) = U(t)$

Therefore $D_{\gamma'} U^{\perp} = D_{\gamma'} U \not\equiv 0$. Otherwise, U is a parallel transport of $U(0) = 0 \Rightarrow U \equiv 0$ (contradiction).

All together, we have proved that

$$D^2\rho^2(v,v) \geq \int_0^b |D_{\gamma'} v^\perp|^2 dt > 0 \qquad (\text{for } v \neq 0)$$

This completes the proof of the thm. ※

The key point of the conclusion of the above thm is that $D^2\rho^2 > 0$ on the whole M, which needs the curvature assumption. Otherwise, we have

Lemma 4: Let
- M = Riem mfd.
- $O \in M$
- $\rho: M \to \mathbb{R}$ distance to O

Then $\exists$ a nbd. $\mathcal{U}_0$ of O in M s.t. ρ^2 is smooth and $D^2\rho^2 > 0$ in $\mathcal{U}_0$.

Sketch of Pf: Let $\mathcal{U}$ be a nbd. of O s.t. $\exists$ normal coordinate system $\{x^1, \cdots, x^n\}$ centered at O. Using this, we can show that $v, w \in T_O M$

$$D^2\rho^2(v,w) = 2\langle v, w\rangle \qquad (\text{Ex!})$$

Therefore, at the center O, $D^2\rho^2 > 0$

$\Rightarrow$ $D^2\rho^2 > 0$ in a nbd $\mathcal{U}_0 \subset \mathcal{U}$ of O $\#$

<u>Def</u>: A function $f: M \to \mathbb{R}$ (M = Riem. mfd) is said to be <u>convex</u> (<u>strictly convex</u>)

$\Leftrightarrow$ $\forall$ geodesic γ in M, $f \circ \gamma$ is convex (strictly convex)

- Therefore, a C^∞ $f: M \to \mathbb{R}$ is convex (strictly convex)

$\Leftrightarrow$ $D^2 f \geq 0$ (>0) (Ex!)

<u>Def</u>: Let M = complete Riem. mfd. Then

- a subset $\Omega \subset M$ is called <u>convex</u>

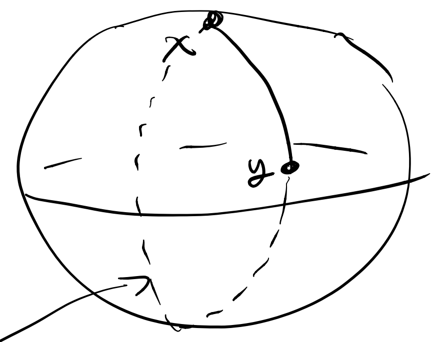
  $\Leftrightarrow$ $\forall$ $x, y \in \Omega$, the <u>shortest geodesic</u> joining x to y is contained in Ω.

- a subset $\Omega \subset M$ is called <u>totally convex</u>

  $\Leftrightarrow$ $\forall$ $x, y \in \Omega$, any geodesic joining x to y is contained in Ω.

Eg 1: On $S^2 \subset \mathbb{R}^3$, geodesic ball of radius $r < \frac{\pi}{2}$ is convex, but not totally convex

geodesic joining x to y, not contained in $B_r(x)$



Furthermore, geodesic ball of radius r between $\frac{\pi}{2}$ & π is not even convex. (Ex!)

Note: If M is a simply-connected complete Riem. mfd with nonpositive sectional curvature. Then Cartan-Hadamard $\Rightarrow$ any geodesic is minimizing. Therefore, a convex subset of M is also totally convex.

Eg 2: Cylinder $\{x^2+y^2=1\} \subset \mathbb{R}^3$.

Then B_r is convex for $r \leq \frac{\pi}{2}$,

not convex for $r > \frac{\pi}{2}$.

<u>Lemma 5</u>: Let M = Riem. mfd.

(1) Let • $\tau : M \to \mathbb{R}$ is a convex function

• $M_c \overset{\text{def}}{=} \{x \in M : \tau(x) < c\}$ be the sublevel set

• $\gamma : [a,b] \to M$ be a geodesic

Then $\gamma(a), \gamma(b) \in M_c \Rightarrow \gamma([a,b]) \subset M_c$.

(2) Furthermore, if M is complete, then M_c is totally convex.

<u>Pf</u>: (1) $\tau \circ \gamma(t) \leq \max\{\tau\circ\gamma(a), \tau\circ\gamma(b)\} < c$

since $\tau\circ\gamma$ convex

(2) Easily follows from (1).

Cor (of Thm 3) Geodesic balls of a simply-connected complete Riem. mfd M with nonpositive sectional curvature are totally convex.

In particular, $\forall x \in M$, $\{x\}$ is totally convex.
Therefore, there is no nontrivial geodesic $\gamma : [a,b] \to M$ s.t. $\gamma(a) = \gamma(b) = x$.

Thm 6 (J.H.C. Whitehead) Let M = Riem. mfd.
Then $\forall x \in M$, $\exists$ a convex nbd. of x.

Pf: $\forall x \in M$, Lemma 4 (& properties of $\exp_x$)

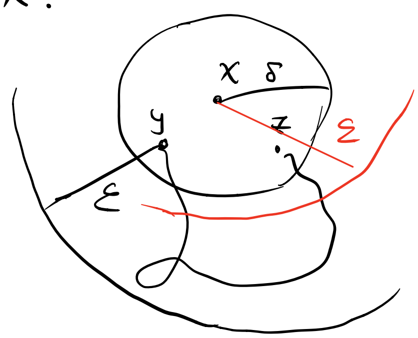
$\Rightarrow \exists \varepsilon > 0$ s.t.

$$\begin{cases} \bullet\ \exp_x : \underset{\subset T_xM}{B(\varepsilon)} \to \underset{\subset M}{B_\varepsilon(x)} \text{ is a diffeomorphism} \\ \bullet\ B_\varepsilon(x) = \exp_x(B(\varepsilon)) \text{ has compact closure in } M \\ \quad \text{(note: } M \text{ may not be complete)} \\ \bullet\ \rho^2 \text{ is } C^\infty \ \&\ D^2\rho^2 > 0 \text{ on } B_\varepsilon(x) \\ \quad \text{where } \rho = \text{distance to } x. \end{cases}$$

In fact, by choosing a smaller $\varepsilon > 0$, we can also assume that $\forall\, y \in B_\varepsilon(x)$, $\exp_y\big|_{B(\varepsilon)}$ is a diffeomaphism.

Let $\delta = \frac{\varepsilon}{3} > 0$ and the geodesic ball $B_\delta(x)$.

We claim that $B_\delta(x)$ is convex.



$\forall$ fixed $y \in B_\delta(x)$,

we observe that $B_\delta(x) \subset B_\varepsilon(y)$.

In fact, $\forall\, z \in B_\delta(x)$,

$$d(z,y) \leq d(z,x) + d(x,y) \leq \delta + \delta = 2\delta = \frac{2\varepsilon}{3} < \varepsilon$$

$\Rightarrow \quad B_\delta(x) \subset B_\varepsilon(y)$.

Therefore, $\forall\, z \in B_\delta(x)$, $\exists$ shortest geodesic ζ joining z to y with $\zeta \subset B_\varepsilon(y)$ and $L(\zeta) < \varepsilon$.

If $\zeta \not\subset B_\varepsilon(x)$, then $y, z \in B_\delta(x) \Rightarrow$

$$L(\zeta) > 2(\varepsilon - \delta) = \frac{4\varepsilon}{3} > \varepsilon \quad \text{contradiction}$$

$\Rightarrow \quad \zeta \subset B_\varepsilon(x)$

Since $D^2\rho^2 > 0$ on $B_\varepsilon(x)$, statement (1) of Lemma 5 on $B_\delta(X)$ ($\subset B_\varepsilon(X)$) $\Rightarrow$

$B_\delta(X) =$ sublevel set of ρ^2

$\Rightarrow$ $\xi \subset B_\delta(X)$ since ξ is the shortest geodesic joining z to y.

Since $y \in B_\delta(X)$ is arbitrary, we've shown that $\forall\, y, z \in B_\delta(x)$, $\exists$ shortest geodesic $\xi \subset B_\delta(X)$ joining y and z. $\therefore$ $B_\delta(X)$ is convex. ⋇

Application 2: Synge Thm

Facts: • A C^∞ mfd M of n-dim. is said to be orientable $\Leftrightarrow$ $\exists$ a nowhere zero C^∞ n-form ω on M.

(i.e. $\omega = f\, dx^1 \wedge \cdots \wedge dx^n$ in local coordinates.

Alternating $(0,n)$-tensor:

$$\omega(X_1, \cdots X_i, \cdots, X_j, \cdots, X_n) = -\omega(X_1, \cdots, X_j, \cdots, X_i, \cdots X_n)$$
)

- If such an ω is chosen, then it is called the <u>orientation</u> of M

  $(\omega_1 \sim \omega_2 \Leftrightarrow \omega_1 = f\omega_2$ for some function $f>0)$

- Let ω be a nowhere zero n-form on such an M, then bases of T_xM can be divided into 2-classes:

$$\begin{cases} \bullet \text{ positive oriented} : \omega(e_1, \cdots, e_n) > 0 \\ \bullet \text{ negative oriented} : \omega(e_1, \cdots, e_n) < 0 \end{cases}$$

(wrt ω).

<u>Lemma</u> 7 : Let $\gamma : [a,b] \to M$ be closed curve in an <u>orientable</u> Riem. mfd M such that $x = \gamma(a) = \gamma(b)$.

Then the parallel transport along γ

$$P^\gamma : T_xM \to T_xM \text{ has } \det P^\gamma = +1.$$

(Pf : Ex!)

$(\swarrow \pi_1(M) \neq 1)$

<u>Lemma 8</u> : Let M = <u>non-simply-connected</u> <u>compact</u> Riem. mfd

Then $\exists$ closed curve $\gamma : [0,b] \to M$ (for some $b>0$) such that $L(\gamma) \leq L(\alpha)$ for any piecewise C^∞

closed curve α which is non-homotopic to zero.

(i.e $[\alpha] \neq 1$)

(Pf: Omitted)

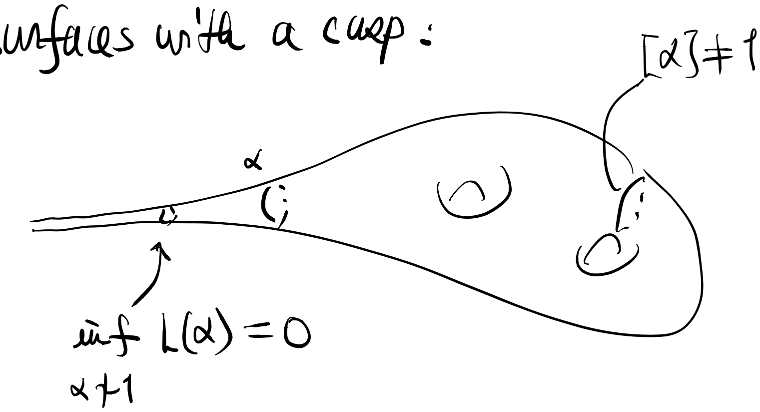
Notes: • $\pi_1(M) \neq 1$ is necessary, otherwise any closed curve can be shrinked to a point

$\Rightarrow \inf_{\gamma} L(\gamma) = 0$

$\Rightarrow$ no closed curves minimizes the length functional.

• Compactness is also necessary:

eg: surfaces with a cusp:



Thm (J.L. Synge) If M is a compact orientable even dim'l Riem mfd with positive sectional curvature, then M is simply-connected.