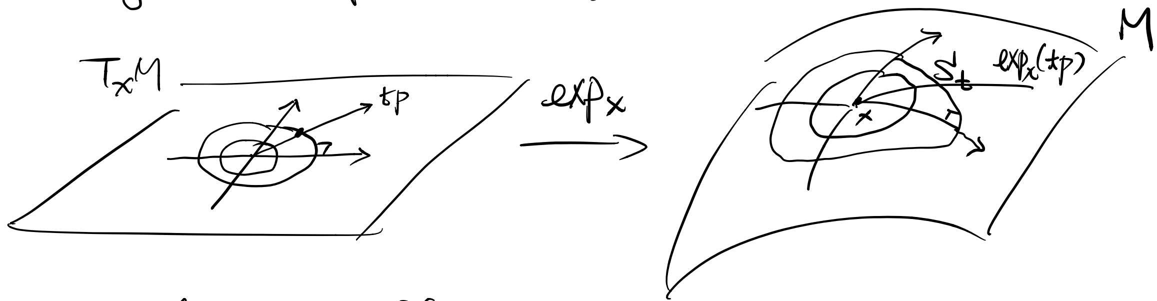
Gauss lemma: Radial geodesics are orthogonal to the geodesic spheres S_t, $\forall t \in (0,\delta)$.



Pf: Define a diffeo

$$F: S^{n-1} \times (0,\delta) \subset T_xM \setminus \{0\} \longrightarrow B_\delta \setminus \{x\}$$
$$(p, t) \longmapsto F(p,t) = \exp_x(tp)$$

Then for fixed $t \in (0,\delta)$

$$F(\cdot, t): S^{n-1} \times \{t\} \to S_t$$

is a diffeomaphism.

Let γ = radial geodesic intersecting S_t at the point $\exp_x(tp)$.

We take a local coordinate $\{y^1, \cdots, y^{n-1}\}$ around $p \in S^{n-1}$.
And let r be the natural parameter of the interval $(0,\delta)$.

Then
$$\begin{cases} R = dF\left(\frac{\partial}{\partial r}\right) \\ Y_i = dF\left(\frac{\partial}{\partial y^i}\right) \quad i = 1, \cdots, n-1 \end{cases}$$

are vector fields on $B_\delta \setminus \{x\} \subset M$ such that

- Y_i are tangential to S_t and form a basis of $T_y S_t$ for $y \in S_t \subset B_\delta \setminus \{x\}$ (in the nbd. of $\exp_x(tp)$)

and

- R is tangential to a radial geodesic.

Therefore, we need to show that

$$\langle R, Y_i \rangle = 0 \ , \ \forall i = 1, \cdots, n-1, \text{ at } \exp_x(tp).$$

derivative wrt the parameter γ.

Then
$$\langle R, Y_i \rangle' = R\langle R, Y_i \rangle$$
$$= \langle D_R R, Y_i \rangle + \langle R, D_R Y_i \rangle$$
$$= 0 + \langle R, D_{Y_i} R \rangle + \langle R, [R, Y_i] \rangle$$

(since $D_R R = D_{\gamma'} \gamma' = 0$)

However
$$[R, Y_i] = [dF(\tfrac{\partial}{\partial r}), dF(\tfrac{\partial}{\partial y_i})]$$
$$= dF([\tfrac{\partial}{\partial r}, \tfrac{\partial}{\partial y_i}]) = 0 .$$

$$\therefore \langle R, Y_i \rangle' = \langle R, D_{Y_i} R \rangle = \tfrac{1}{2} Y_i \langle R, R \rangle$$
$$= 0 \quad (\text{since } \langle R, R \rangle \equiv 1 \text{ by lemma})$$

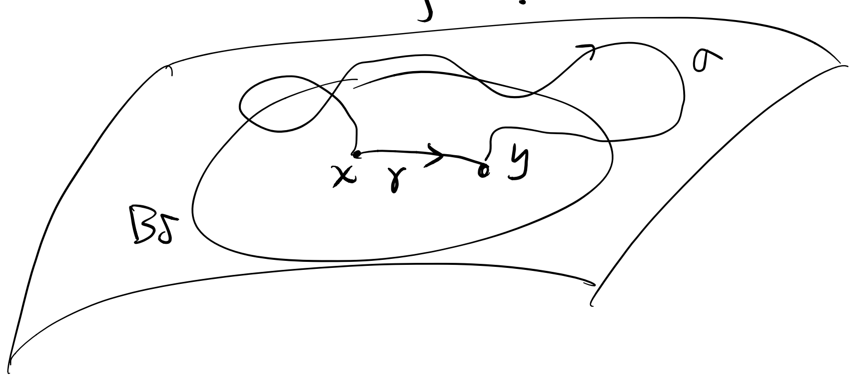
$$\Rightarrow \langle R, Y_i \rangle = \lim_{r \to 0} \langle R, Y_i \rangle (\gamma(r)) = 0$$

since $|Y_i| \to 0$ as $\gamma(r) \to x$ (Ex!) $\#$

<u>Thm</u>: Let
- (M, g) = Riemannian manifold
- $x \in M$
- $\delta > 0$ s.t. $\exp_x : B(\delta) \to B_\delta$ is a diffeo.
- γ = unique radial geodesic joining x and a point $y \in B_\delta \setminus \{x\}$.

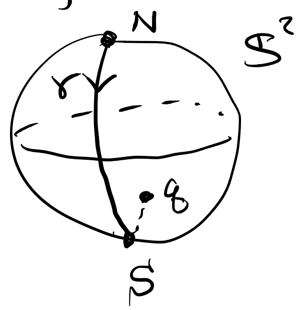
Then $L(\gamma) \leq L(\sigma)$ for all piecewise smooth curve σ on M (not necessary within B_δ) joining x to y.

Equality holds $\Leftrightarrow$ σ = monotonic reparametrization of γ.



<u>Cor</u>: Let $\gamma : [0, c] \to M$ be a arc-length parametrized piecewise smooth curve such that $L(\gamma) \leq L(\sigma)$ for all piecewise smooth curve σ joining $\gamma(0)$ and $\gamma(c)$. Then γ is a geodesic.

Caution: The converse of the Cor. is not ture in general:



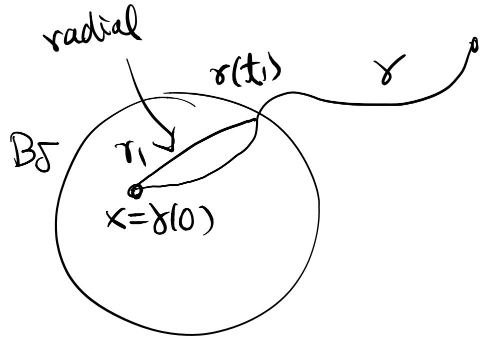
γ = geodesic, but not length minimizing.

Def: A geodesic $\gamma : [0, c] \to M$ is called a <u>minimizing geodesic</u> if $L(\gamma) \le L(\sigma)$ $\forall \sigma$ joining $\gamma(0)$ & $\gamma(c)$ (σ = piecewise smooth)

<u>Pf of Cor</u> (by assuming the Thm)

Let $x = \gamma(0)$. Choose B_δ as in the thm.

Let $t_1 = \min\{t : \gamma(t) \in \partial B_\delta\}$



If t_1 doesn't exist, then we are done.

If t_1 exists, & $\gamma|_{[0,t_1]}$ is not geodesic,

then by the thm

$$L(\gamma|_{[0,t_1]}) > L(\gamma_1)$$

where γ_1 = radial geodesic joining $x=\gamma(0)$ to $\gamma(t_1)$ in $\overline{B_\delta}$ (by continuity)

$\Rightarrow L(\gamma_1 \cup \gamma|_{[t_1, c]}) < L(\gamma)$ which is a contradiction.

Hence $\gamma|_{[0,t]}$ is a geodesic.

Continuing this argument $\Rightarrow \gamma|_{[0,c]}$ is a geodesic #

Pf of the Thm:

As in the proof of the Gauss lemma, we can fixed basis $\{R, Y_1, \cdots, Y_{n-1}\}$ of $T_z M$ for $z \in B_\delta \setminus \{x\}$,

such that

- R = tangential to radial geodesic and $|R| = 1$.

- $Y_1, \cdots, Y_{n-1}$ = tangential to the geodesic spheres.

and $\langle R, Y_i \rangle = 0$, $\forall i = 1, \cdots, n-1$, by Gauss lemma.

We only need to consider $\sigma \in B_\delta$.

For any such $\sigma : [0,1] \to B_\delta$ with

$$\sigma(0) = x, \quad \sigma(1) = y$$

we have $\forall t \in [0,1]$

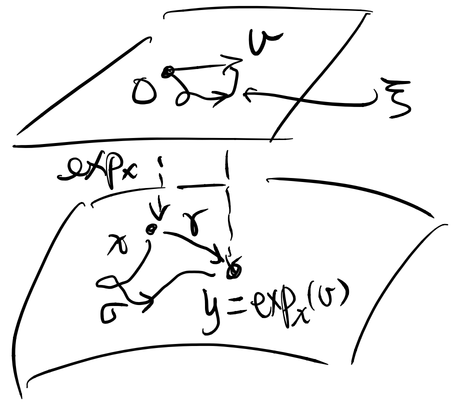
$$\sigma'(t) = f(t) R(\sigma(t)) + T(t)$$

for some function $f(t)$, and

$$T(t) = \text{a linear combination of } Y_i\text{'s}.$$

Let $v \in B(\delta)$ be the unique vector such that

$$\exp_x(v) = y$$



Then

$$\xi = \exp_x^{-1} \circ \sigma$$

is a curve in $B(\delta) \subset T_xM$

Since $(d\exp_x^{-1})(R) = \mathcal{R}$ = radial unit vector field defined before.

$$(d\exp_x^{-1})(Y_i) = \text{tangential to } S^{n-1}_{|\xi(t)|} \subset T_xM,$$

we see that

$$(d\exp_x^{-1})(\langle \sigma', R\rangle R) = f\mathcal{R} \quad \text{by Gauss lemma}$$

is the radial projection of the tangent vector ξ'.

$$\Rightarrow \quad |v| = |\xi(1)| - |\xi(0)| = \int_0^1 f(t)\,dt$$

$\therefore$ the length of the radial geodesic γ joining x to $y = \exp_x(v)$

$$L(\gamma) = |v| = \int_0^1 f(t)\,dt \,.$$

Gauss lemma again $\Rightarrow$

$$|\sigma'(t)|^2 = f^2(t)\,|R(\sigma(t))|^2 + |T(t)|^2$$

$$= f^2(t) + |T(t)|^2$$

$$\Rightarrow \quad L(\sigma) = \int_0^1 |\sigma'| = \int_0^1 \sqrt{f^2+|T|^2} \geq \int_0^1 f(t)\,dt = L(v).$$

Finally, if $L(\sigma) = L(\gamma)$, then $T(t) = 0$ & $f > 0$.

$$\Rightarrow \quad \sigma'(t) = f(t)\,R(\sigma(t)) \text{ with } f > 0$$

$\Rightarrow \quad \sigma =$ monotonic reparametrization of γ. $\#$

§4.3 Completeness, metric structure

(M, g) = Riemannian manifold (connected)

Def: $d: M \times M \to [0, \infty)$ defined by

$$d(x,y) = \inf_{\gamma} L(\gamma).$$

where "inf" is taken over all piecewise smooth curves γ joining x and y, is called the distance (metric) of (M, g).

Thm: (M, d) is a metric space, i.e. d satisfies

(1) $d(x,y) \geq 0$; "=" iff "$x=y$".

(2) $d(x,y) = d(y,x)$,

(3) $d(x,y) \leq d(x,z) + d(z,y)$.

Pf: All are easy (Ex.) and we prove only

$$\text{"} d(x,y) = 0 \Rightarrow x = y \text{"}.$$

Suppose $x \neq y$.

If $y \in B_\delta$, where δ is given as in the Thm in §4.2, then $d(x,y) = L(\gamma)$ where γ = radial geodesic from x to y

$\Rightarrow \quad d(x,y) > 0$.

Continuity argument $\Rightarrow d(x,y) = \delta > 0$, if $y \in \partial B_\delta$.

Hence if $y \notin B_\delta$, and σ = curve joining x to y.

Choose the 1st point y_1 of σ on ∂B_σ and conclude that

$$L(\sigma) \geq L(\sigma|_{[\text{from } x \text{ to } y_1]})$$
$$\geq \delta > 0.$$

Taking "inf" $\Rightarrow d(x,y) \geq \delta > 0$. #

In fact, we have a stronger theorem

<u>Thm</u>: The topology of (M,d) is the same as the original topology of M.

(Pf: Ex, or pg 61-62 of Wu, or doCarmo.)

<u>Def</u>: A Riemannian manifold (M,g) is said to be <u>complete</u> if the associated metric space (M,d) is complete.

<u>egs</u>: $(\mathbb{R}^n$, standard metric$)$, $(S^n$, standard metric$)$ are

complete.

Hopf-Rinow Theorem: The following statements are equivalent on a Riemannian manifold (M, g):

(1) M is complete

(2) $\forall x \in M$, $\exp_x$ defined on the whole T_xM,

(3) $\exists x \in M$, $\exp_x$ defined on the whole T_xM,

(4) bounded closed subsets of M are compact.

Cor 1 of Hopf-Rinow Thm.

If (M, g) is complete, then $\forall x \neq y \in M$, $\exists$ a minimizing geodesic γ joining x and y.

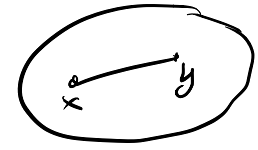
(Recall: all manifolds are connected in this course!)

Cor 2: If (M, g) is complete, then $\forall x \in M$

$$\exp_x : T_xM \to M \text{ is surjective.}$$

Notes: • The converse of Cor 1 of Hopf-Rinow is not true in general:

eg: convex subsets in $\mathbb{R}^n$:



- A general complete metric space may <u>not</u> have <u>Heine-Borel property</u>.

<u>eg</u>: $S = \{a_1, a_2, \cdots\}$ countable infinite set of distinct elements with <u>discrete metric</u> d on S, ie.

$$d(a_i, a_j) = 1 - \delta_{ij}.$$

Then (S, d) is a complete metric space which is bounded

$\Rightarrow$ S is a closed and bounded set, but not <u>compact</u>.

<u>Pf of Hopf-Rinow Thm</u>

(1)$\Rightarrow$(2) Let $\gamma : [0, \delta) \to M$ be a geodesic

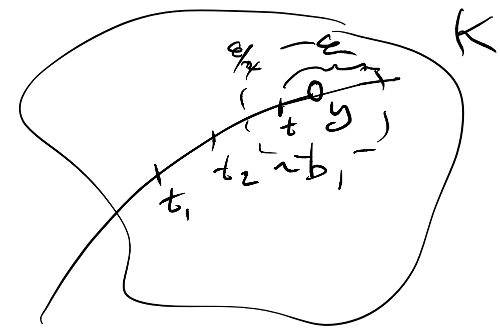
$$\gamma(t) = \exp_x(tv) \text{ for some } v \in T_xM \ (|v|=1)$$

Suppose that $I = (a_1, b_1)$ is the maximal possible interval containing $[0, \delta)$ such that $\gamma(t)$ is defined.

Suppose $b_1 < +\infty$. Then M complete

$\Rightarrow \exists y \in M$ s.t. $\lim_{t \to b_1} \gamma(t) = y$

Let K = cpt. nbd. of y



ODE theory $\Rightarrow \exists\, \varepsilon > 0$

indep. of t s.t.

" If $d(\gamma(t), y) < \frac{\varepsilon}{2}$ ($\&$ $\gamma(t) \in K$)

then $\exists$ minimizing geodesic $\zeta = [0, \varepsilon] \to M$

s.t. $\zeta(0) = \gamma(t)$ & $\zeta'(0) = \gamma'(t)$. "

$\Rightarrow$ joining ζ to γ gives an extension of γ

beyond b_1. Hence $b_1 = +\infty$.

Similar argument $\Rightarrow$ $a_1 = -\infty$.

$\therefore$ $\exp_x(tv)$ defined $\forall\, t \in (-\infty, \infty)$

Since v (with $|v| = 1$) is arbitrary, $\exp_x$ defined on the whole T_xM.

Since $x \in M$ is arbitrary, this implies (2).

(2) $\Rightarrow$ (3) trivial.

(4) $\Rightarrow$ (1) standard fact in metric space theory.

To prove (3) $\Rightarrow$ (4), we claim that

(3) $\Rightarrow$ (5), where

(5) Assume $x \in M$ as in (3), then $\forall y \in M$, $\exists$ a minimizing geodesic joining x to y.

Pf of (3) $\Rightarrow$ (5)

Let $\overline{\mathcal{B}}(r) = \{ y \in M : d(x,y) \le r \}$

$$\overline{\mathcal{B}}(r) \supset \Sigma(r) = \{ y \in \overline{\mathcal{B}}(r) : y \text{ is joined to } x \text{ by a min. geodesic} \}.$$

Then we need to show that

$$\overline{\mathcal{B}}(r) = \Sigma(r), \quad \forall r \in [0, \infty)$$

Let $\mathcal{J} = \{ r \in [0,\infty) : \overline{\mathcal{B}}(r) = \Sigma(r) \}$.

(i) We have shown that if $r < \delta$, where $\delta > 0$ is given by the Thm in §4.2 then $r \in \mathcal{J} \Rightarrow \mathcal{J} \neq \emptyset$.

(ii) Since $\exp_x$ defined on whole $T_xM \cong \mathbb{R}^n$, and $\exp_x(tv)$ continuously depends on $v \Rightarrow \mathcal{J}$ is closed.

(ii) To show $\mathcal{I}$ is open, we need the following fact (Ex, or see doCarmo)

$$(*)\left[\begin{array}{l} \forall \text{ cpt } K \subset M,\ \exists \varepsilon > 0 \text{ such that} \\ \quad \forall\, y, z \in K \text{ with } d(y,z) \le \varepsilon, \\ \text{then } \exists \text{ a minimizing geodesic joining } y \,\&\, z. \end{array}\right.$$

Note: This is a stronger result then the Thm in §4.2 in which one of the points has to be the center.

Pf of openess: let $r \in \mathcal{I}$, then $\overline{\mathcal{B}}(r) = \Sigma(r)$

$\Rightarrow \overline{\mathcal{B}}(r) \subset \exp_x(\overline{B(r)}) \Rightarrow \overline{\mathcal{B}}(r)$ is compact

$\Rightarrow \partial\overline{\mathcal{B}}(r)$ is also compact

Now $\forall\, z \in \partial\overline{\mathcal{B}}(r)$, $\exists\, \varepsilon_1(z) > 0$ s.t.

$\exp_z : B(\varepsilon_1) \to B_{\varepsilon_1}(z)$ is a diffeo.

By compactness of $\partial\overline{\mathcal{B}}(r)$, $\exists$ finitely many z_i s.t.

$\{B_{\frac{1}{2}\varepsilon_1(z_i)}(z_i)\}$ covers $\partial\overline{\mathcal{B}}(r)$. This implies

$K = \overline{\mathcal{B}}(r) \cup \left(\bigcup_i \overline{B_{\frac{1}{2}\varepsilon_1(z_i)}(z_i)} \right)$ is cpt containing $\overline{\mathcal{B}}(r+\varepsilon_2)$ for some $0 < \varepsilon_2 < \frac{1}{2}\varepsilon_1(z_i)$, $\forall i$.

Applying $(*)$, $\exists \varepsilon > 0$ with property in $(*)$.

Let $\varepsilon' \in (0, \min\{\varepsilon_2, \varepsilon\})$ and consider

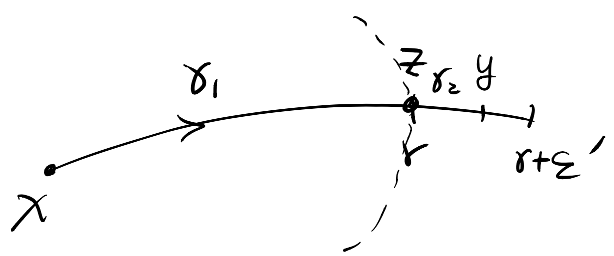
$$y \in \overline{B}(r+\varepsilon').$$

If $y \in \overline{B}(r)$, then $y \in \Sigma(r) \subset \Sigma(r+\varepsilon')$.

If $y \in \overline{B}(r+\varepsilon') \setminus \overline{B}(r)$, then $\exists\, z \in \partial \overline{B}(r)$ s.t.

$$d(x,y) = d(x,z) + d(z,y)$$

(by using the cptness of $\partial\overline{B}(r)$ & definition of $d(x,y)$)



Then $r \in \mathcal{J} \Rightarrow \exists$ minimizing geodesic γ_1 joining x and z.

On the other hand,

$$d(z,y) = d(x,y) - d(x,z) \leq r + \varepsilon' - r = \varepsilon' < \varepsilon$$

$(*) \Rightarrow \exists$ minimizing geodesic γ_2 joining z & y

Then connecting γ_1 & γ_2, we have a piecewise smooth curve joining x to y with

$$\text{length} = d(x,z) + d(z,y) = d(x,y)$$

$\Rightarrow$ it must be a minimizing geodesic

$\therefore \quad y \in \Sigma(r+\varepsilon')$

Hence $\quad \overline{B}(r+\varepsilon') \subset \Sigma(r+\varepsilon')$

$\therefore$ J is open.

(i), (ii) & (iii) together $\Rightarrow J = [0, \infty)$

$\therefore$ we've proved $(3) \Rightarrow (5)$.

<u>$(3) \Rightarrow (4)$</u>

$\forall$ bounded & closed set K, $\exists A > 0$ s.t.

$$d(x,k) \leq A, \quad \forall k \in K.$$

$\Rightarrow \quad K \subset \exp_x(\overline{B}(A))$ (by assumption (3))

$\Rightarrow \quad K$ is cpt. (since $\overline{B}(A)$ is cpt).

This completes the proof of Hopf-Rinow theorem.

Pf of Cor 1 : Hopf-Rinow $\Rightarrow$ (2) is true

$(\Rightarrow$ (3) is true$)$

$\Rightarrow$ (5) is true $\forall x \in M$

$\Rightarrow$ Cor 1 is true. #