Lemma 4 (Ricci Identity)

$$\boxed{D^2T(\cdots,X,Y) - D^2T(\cdots,Y,X) = (R_{XY}T)(\cdots)}$$

$\forall$ tensor field T;

$X, Y \in \Gamma(TM)$.

Note: So we usual denote Ricci Identity by

$$\boxed{R_{XY} = D^2_{XY} - D^2_{YX}}$$

Pf: $D^2T(\cdots,X,Y)$

$= D(DT)(\cdots,X,Y)$

$= (D_Y(DT))(\cdots,X)$

$= D_Y[(DT)(\cdots,X)] - \sum DT(\cdots D_Y\cdot,\cdots,X) - DT(\cdots,D_YX)$

$= D_Y[(D_XT)(\cdots)] - \sum (D_XT)(\cdots D_Y\cdot\cdots) - (D_{D_YX}T)(\cdots)$

$= (D_YD_XT)(\cdots) - (D_{D_YX}T)(\cdots)$

$= (D_YD_XT - D_{D_YX}T)(\cdots)$

Similarly for $D^2T(\cdots, Y, X)$

$$= \left(D_X D_Y T - D_{D_X Y} T\right)(\cdots)$$

Hence

$$D^2T(\cdots, X, Y) - D^2T(\cdots, Y, X)$$

$$= \left(D_Y D_X T - D_{D_Y X} T - D_X D_Y T + D_{D_X Y} T\right)(\cdots)$$

$$= \left(-D_X D_Y T + D_Y D_X T + D_{[X,Y]} T\right)(\cdots)$$

using torsion free condition

$$= \left(R_{XY} T\right)(\cdots) \;.$$

3.4 Various notions of curvature

Def: The Ricci tensor "Ric" is the $(0,2)$-tensor field defined by

$$\mathrm{Ric}(X,Y) = \sum_{i=1}^{n} R(e_i, X, e_i, Y), \quad \forall X, Y \in \Gamma(TM)$$

where $\{e_i\}$ = orthonormal basis for T_xM.

Note: • Ric doesn't depend on the o.n. basis $\{e_i\}$. (Ex!)

• Ric is symmetric, i.e. $Ric(X,Y) = Ric(Y,X)$.

Def: Let $X \in T_xM$ with $|X| = 1$. Then $Ric(X,X)$ is called the Ricci curvature in the direction of X.

Note: One can choose an o.n basis $\{e_1, \dots, e_n\}$ of T_xM such that $e_1 = X$. Then by definition of Ric:

$$
\begin{aligned}
Ric(X) \overset{def}{=} Ric(X,X) &= \sum_{i=1}^{n} R(e_i, X, e_i, X) \\
&= \sum_{i=1}^{n} R(e_i, e_1, e_i, e_1) \\
&= \sum_{i=2}^{n} R(e_i, e_1, e_i, e_1) \\
&= \sum_{i=2}^{n} K(\Pi_i) \quad \text{where } \Pi_i = \text{span}\{e_1, e_i\}
\end{aligned}
$$

Def: The scalar curvature $S(x)$ at $x \in M$ is defined by

$$
S(x) = \sum_{i,j} R(e_i, e_j, e_i, e_j)
$$

where $\{e_i\}$ = o.n. basis for T_xM.

i.e. scalar curvature = "sum of all sectional curvatures of planes given by an o.n. basis"

= "sum of Ric curvatures given by an o.n. basis."

Ch4 Exponent Map, Gauss Lemma, & Completeness

Let
- M = Riemannian manifold with metric
- $g = g_{ij}\, dx^i \otimes dx^j \quad (g = \langle\, ,\, \rangle)$
- D = Levi-Civita connection of g.

4.1 Exponent map

Recall: $\gamma : [0, L] \to M$ is a geodesic (wrt D)

$$\iff D_{\gamma'} \gamma' = 0.$$

Facts:
- If γ is a geodesic, $|\gamma'|$ is a constant.
- If $\gamma : [0, L] \to M$ is a geodesic, then $\forall$ constant $c > 0$,

$$\gamma^c : [0, \tfrac{L}{c}] \to M$$
$$t \mapsto \gamma(ct)$$

is also a geodesic, and

$$|(\gamma^c)'| = c|\gamma'|$$

Therefore, we can normalize our geodesic to have

$$|\gamma'| = 1.$$

Recall: If $\xi: [a,b] \to M$ is a C^∞ curve, then the length of ξ is defined by

$$L(\xi) = \int_a^b |\xi'| \, dt.$$

If ξ is regular, i.e. $|\xi'(t)| > 0$, $\forall t \in [a,b]$,

then $$s(t) = \int_a^t |\xi'(\tau)| \, d\tau = L(\xi|_{[a,t]})$$

defines a C^∞ function $s: [a,b] \to [0, L(\xi)]$

with $\frac{ds}{dt} = |\xi'(t)| > 0$.

Hence $u = s^{-1}: [0, L(\xi)] \to [a,b]$ exists & C^∞.

$\Rightarrow$ $\tilde{\xi}(s) \overset{\text{def}}{=\!=} \xi(u(s)): [0, L(\xi)] \to M$ is a reparametrization of ξ such that $\left|\frac{d\tilde{\xi}}{ds}\right| = 1$.

Terminology:

- s = arc-length parameter.
- $\tilde{\xi}$ is said to be parametrized by arc-length.
- a normalized geodesic is a geodesic parametrized by arc-length,

i.e. $D_{\gamma'}\gamma' = 0$ and $|\gamma'| = 1$.

<u>Note</u>: All the above can be extended to piecewise C^1 curves.

<u>Recall</u>: $D_{\gamma'}\gamma' = 0$ is a (nonlinear) ODE system and hence we have the following result by "applying" the theory of ODE:

<u>Thm</u>: $\forall\, x \in M$ & $\varepsilon > 0$,

$\exists$ nbd. $\mathcal{U}$ of x, and $\delta > 0$ such that

$\forall\, y \in \mathcal{U}$ and $v \in T_y M$ with $|v| < \delta$,
$\exists$ unique geodesic $\gamma_v : I \to M$, defined on an open interval I containing $[-\varepsilon, \varepsilon]$, with initial condition

$$\begin{cases} \gamma_v(0) = y \\ \gamma_v'(0) = v \end{cases}$$

If γ_v is a geodesic by above, then

$$\xi_v(t) \overset{\text{def}}{=\!=} \gamma_v(\varepsilon t) \text{ is a geodesic}$$

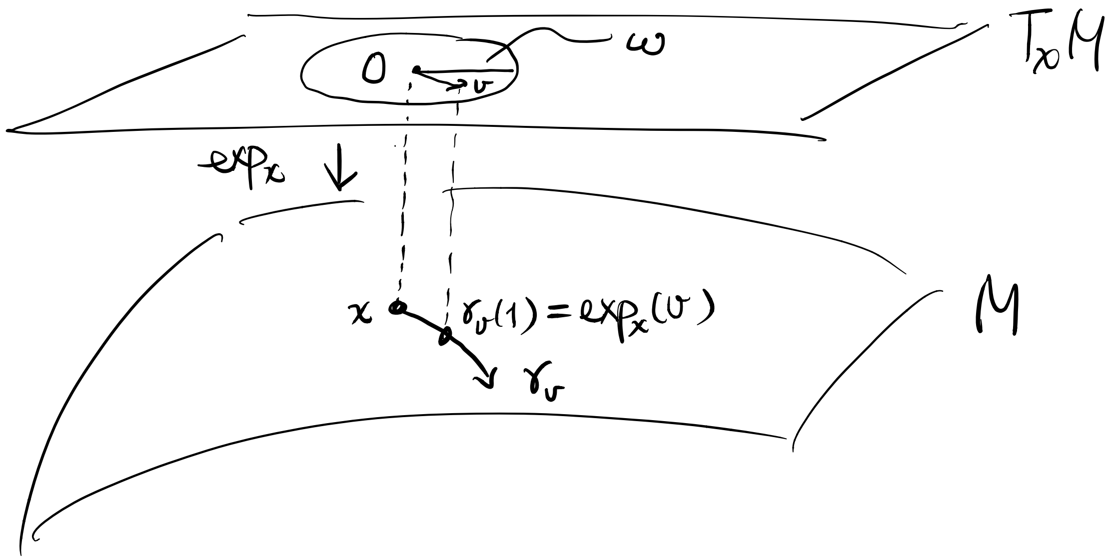
defined on an open interval containing $[0,1]$.

<u>Thm(#)</u> $\forall\ x\in M$, $\exists$ nbd $\mathcal{U}$ of x and $\omega>0$ s.t.

$\forall\ y\in\mathcal{U}$ and $v\in T_yM$ with $|v|<\omega$,
$\exists$ unique geodesic $\gamma_v: I\to M$ defined on an open interval I containing $[0,1]$ with initial condition
$$\begin{cases}\gamma_v(0)=y\\ \gamma_v'(0)=v.\end{cases}$$

<u>Def</u>: Let $\omega>0$ be given in Thm(#). The <u>exponential map</u> $\exp_x$ at x, defined on
$$B_x(\omega)=\{v\in T_xM : |v|<\omega\}\subset T_xM$$
is the map
$$\begin{aligned}\exp_x : B_x(\omega) &\longrightarrow M\\ \rotatebox{90}{$\in$}\quad &\qquad \rotatebox{90}{$\in$}\\ v &\longmapsto \gamma_v(1)\end{aligned}$$
where γ_v is given by Thm(#)

This is $\boxed{\exp_x(v)=\gamma_v(1)}$



Fact: Let $\mathcal{V} = \{(y,v) \in TM : y \in \mathcal{U},\ |v| < \omega\} \subset TM$ (with $\mathcal{U}$ & $\omega > 0$ as in Thm(#)). Then Thm(#)

$\Rightarrow$ $\exp(y,v) \overset{\text{def}}{=\!=} \exp_y(v)$ defines a map

$$\exp : \mathcal{V} \to M.$$

By ODE theory & Thm(#), $\exp : \mathcal{V} \to M$ is C^∞

& in particular

$$\exp_x : B_x(\omega) \to M \text{ is } C^\infty.$$

(Pf: See Gallot, Hulin & Lafontaine)

Note: In fact, we can show that

$$\exp_x : \mathcal{B} \to M \in C^\infty$$

on the maximal domain of the definition of $\exp_x$.

Note: In the case of

$$M = SO(n, \mathbb{R}) = \{A \in n\times n\text{-matrix} : A^T A = I,\ \det A = 1\}$$

with metric defined by $(n-2)\operatorname{tr}(XY)$ for

$$X, Y \in so(n, \mathbb{R}) = T_{Id}M = \{B \in n\times n\text{-matrix} : B^T + B = 0\}.$$

Then

$\exp_{Id} : T_{Id}M \to M$ is given by

$$\exp_{Id} B = e^B = \sum_{k=0}^{\infty} \frac{B^k}{k!}$$

$$\forall\, B \in T_{Id}M = \{B^T + B = 0\} = so(n, \mathbb{R}).$$

This is the reason for the terminology.

Thm: $\exp_x$ is a diffeomorphism in a nbd of $0 \in T_xM$.

This Thm follows immediately from

Lemma: $\boxed{(d\exp_x)_0 = \text{"identity of } T_xM\text{"}}$

Note :

$$\exp_x : \underset{\subset T_xM}{B(\omega)} \longrightarrow M$$

with $\exp_x(0) = x$

$$\Rightarrow \quad (d\exp_x)_0 : T_0(T_xM) \longrightarrow T_xM$$

Since T_xM is linear, $T_0(T_xM) \cong T_xM$

In fact, $\forall\ v \in T_xM$, we define

$$\xi_v : t \mapsto tv \quad \text{a curve in } T_xM$$

Note that $\xi_v(0) = 0$, $\xi_v'(0) = v$

$\therefore$ ξ_v represents a tangent vector of T_xM at 0.

i.e. $[\xi_v] \in T_0(T_xM)$.

One can check that this is an identification of

$$T_0(T_xM) \cong T_xM.$$

Hence $(d\exp_x)_0$ can be regarded as a map from T_xM to itself under the identification.

Pf: $\forall\, v \in T_xM \cong T_0(T_xM)$ (ie $v \Leftrightarrow [t \mapsto tv]$)

$$(d\exp_x)_0(v) = \frac{d}{dt}\Big|_{t=0} \exp_x(tv)$$
$$= \frac{d}{dt}\Big|_{t=0} \gamma_{tv}(1)$$
$$= \frac{d}{dt}\Big|_{t=0} \gamma_v(t) \leftarrow \left(\begin{array}{c}\text{by scaling}\\ \text{property of}\\ \text{geodesic}\end{array}\right)$$
$$= \gamma_v'(0) = v.$$

※

We can even prove a stronger result:

Thm: $\forall$ compact $K \subset M$, $\exists\, \varepsilon > 0$ s.t.

$\forall\, x \in K$, $\exp_x$ is deffeo. on $B_\varepsilon(x) = \{v \in T_xM : |v| < \varepsilon\}$

(That is, for fixed cpt K, we can find a uniform $\varepsilon > 0$)

Pf: It is sufficient to show that

$$\boxed{\begin{array}{l}\forall\, x \in M, \exists\, \varepsilon > 0, \text{ \& open nbd } \Omega \text{ of } x \text{ s.t.}\\ \forall\, y \in \Omega, \exp_y \text{ is a diffeo. on } B_y(\varepsilon) \subset T_yM\end{array}}$$

By Thm(#), $\exists$ nbd. $\mathcal{U}$ of x s.t. $\exp_y$ is defined on some ball $B_y(\varepsilon(y))$, $\varepsilon(y)>0$.

Let $N=\{(y,v): y\in\mathcal{U},\ v\in B_y(\varepsilon(y))\}\subset TM$

and define

$$\begin{aligned} E = N &\longrightarrow M\times M \\ \psi\quad & \qquad\quad \psi \\ (y,v) &\longmapsto (y, \exp_y v) \end{aligned}$$

By the theory of ODE, E is C^∞.

Choose a coordinate system $\{x^1,\cdots,x^n\}$ centered at x (i.e. $x^i(x)=0$, $\forall i=1,\cdots,n$). Then for any (y,v), we can represent it by coordinates

$$(x^1,\cdots,x^n,u^1,\cdots,u^n)$$

where $\{u^i\}$ are given by $v=\sum u^i\frac{\partial}{\partial x^i}$.

(Note: $u^i = dx^i(v)$, $\forall i=1,\cdots,n$)

$\Rightarrow$ $\left\{\frac{\partial}{\partial x^1},\cdots,\frac{\partial}{\partial x^n},\frac{\partial}{\partial u^1},\cdots,\frac{\partial}{\partial u^n}\right\}$ is a basis of the tangent space $T_{(y,v)}(TM)$ of TM.

Now
$$dE_{(x,0)}\left(\frac{\partial}{\partial x^i}\Big|_{(x,0)}\right) = \frac{d}{dt}\Big|_{t=0} E(\xi_i(t), 0)$$

where $\xi_i(t)$ is a curve in M s.t.
$$\xi_i(0) = x \quad \& \quad \xi_i'(0) = \frac{\partial}{\partial x^i}\Big|_x$$

$$\therefore \quad dE_{(x,0)}\left(\frac{\partial}{\partial x^i}\Big|_{(x,0)}\right) = \frac{d}{dt}\Big|_{t=0}\left(\xi_i(t), \exp_{\xi_i(t)}(0)\right)$$
$$= \frac{d}{dt}\Big|_{t=0}\left(\xi_i(t), \xi_i(t)\right)$$
$$= \left(\frac{\partial}{\partial x^i}\Big|_x, \frac{\partial}{\partial x^i}\Big|_x\right)$$

Also
$$dE_{(x,0)}\left(\frac{\partial}{\partial u^i}\Big|_{(x,0)}\right) = \frac{d}{dt}\Big|_{t=0} E\left(x, t\frac{\partial}{\partial x^i}\Big|_x\right)$$
$$= \frac{d}{dt}\Big|_{t=0}\left(x, \exp_x\left(t\frac{\partial}{\partial x^i}\Big|_x\right)\right)$$
$$= \left(0, (d\exp_x)_0\left(\frac{\partial}{\partial x^i}\Big|_x\right)\right)$$
$$= \left(0, \frac{\partial}{\partial x^i}\Big|_x\right) \quad \text{(by previous lemma)}$$

$\Rightarrow \quad dE_{(x,0)} : T_{(x,0)}N \to T_xM \times T_xM$ is nonsingular

$\therefore$ IFT $\Rightarrow$ E is a local diffeo that maps a nbd $\mathcal{W}$ of $(x,0)$ in TM to a nbd of $(x, \exp_x(0)) = (x,x)$ in $M \times M$.

Therefore, $\exists\, c > 0$, $\varepsilon' > 0$ such that

$$\{(y,v) \in TM : |x^i(y)| \le c,\ |u^i(v)| \le \varepsilon'\}$$

is a cpt subset of $\mathcal{W}$.

$\Rightarrow \exists\, \varepsilon > 0$ s.t.

norm wrt g

$$\{(y,v) \in TM : |x^i(y)| \le c,\ |v| \le \varepsilon\} \subset \mathcal{W}.$$

Then this $\varepsilon > 0$ & $\Omega = \{y \in \mathcal{U} : |x^i(y)| \le c\}$ satisfying the requirement. ※

§4.2 Gauss Lemma, minimizing geodesic

Let (M,g) = Riemannian manifold,

$x \in M$, & $\delta > 0$ sufficiently small s.t.

$\exp_x : B(\delta) \to B_\delta$ is a diffeomorphism,

where $B(\delta) = \{ v \in T_xM : |v| < \delta \}$ $\quad (|v| = \langle v, v \rangle^{1/2})$

$B_\delta = \exp_x (B(\delta))$

Then • $\gamma(t) = \exp_x(tv)$, $t \in (0, 1]$, $v \in B(\delta)$

is called a radial geodesic (segment) joining x to $\exp_x v$.

And $\forall\, t \in (0, \delta)$,

• $S_t = \exp_x \left(\{ v \in T_xM : |v| = t \} \right)$ is called the geodesic sphere of radius t centered at x.

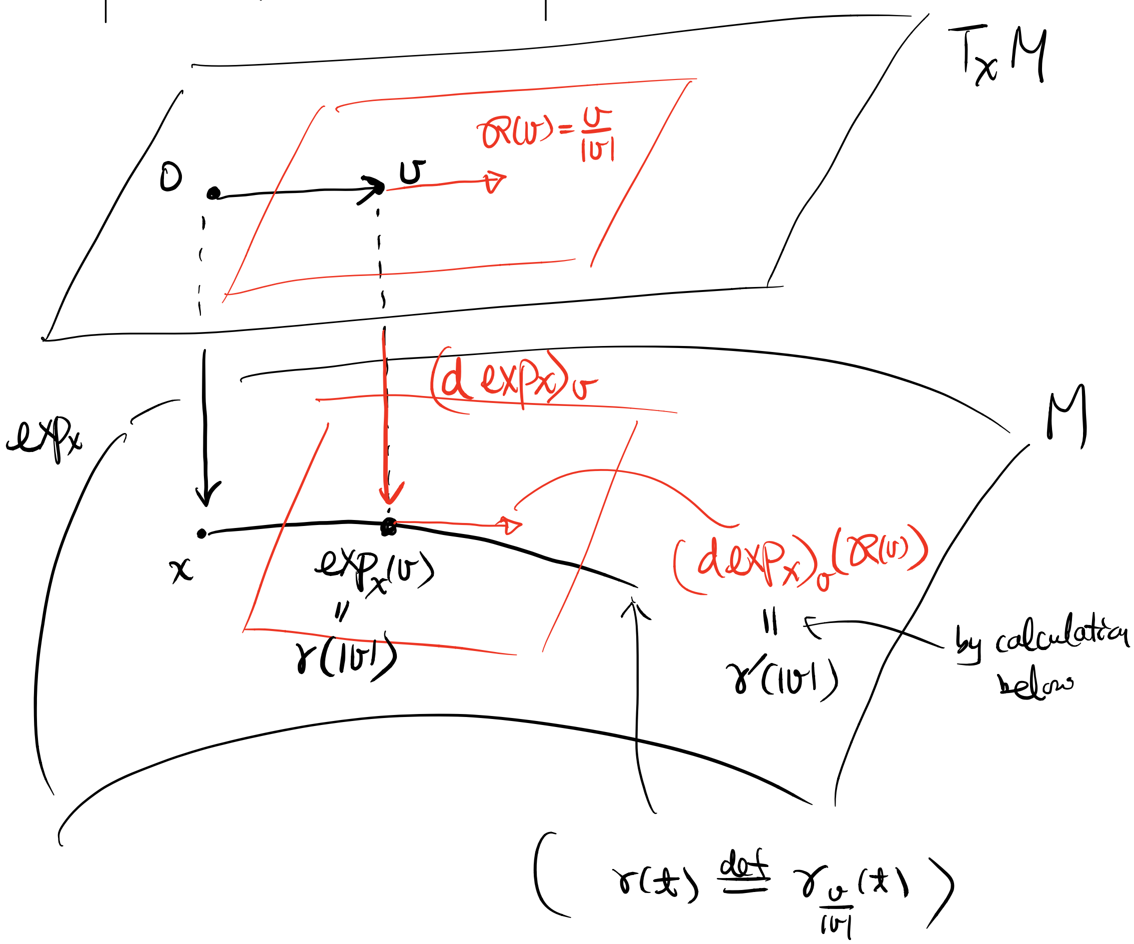
• $B_t = \exp_x (B(t)) = \exp_x \left(\{ v \in T_xM : |v| < t \} \right)$ is called the geodesic ball of radius t centered at x.

Lemma: (M, g), x, δ as above. Define a vector field $\mathcal{R}$ on $T_xM \setminus \{0\}$ by

$$\mathcal{R}(v) = \frac{v}{|v|} \qquad \left(\begin{array}{l} \mathcal{R} : T_xM\setminus\{0\} \to T(T_xM\setminus\{0\}) \\ \quad\quad\;\; \overset{\cup\!\!\shortmid}{v} \end{array} \right)$$

with $T_v(T_xM\setminus\{0\}) \equiv T_xM$

Then $$\left| (d\exp_x)_v (\mathcal{R}(v)) \right| = 1 \qquad (\text{for } v \in B(\delta))$$



Pf: For $v \in B(\delta) \setminus \{0\} \subset T_x M \setminus \{0\}$, let

$\gamma(t) = \gamma_{\frac{v}{|v|}}(t)$ the normalized geodesic on M

s.t.

$$\begin{cases} \gamma(0) = x \\ \gamma'(0) = \frac{v}{|v|} \end{cases} \quad \left(\text{i.e. } \gamma(t) = \exp_x\left(t\frac{v}{|v|}\right)\right)$$

By definition of $\exp_x$,

$$\exp_x(v) = \gamma(|v|).$$

Since $\mathcal{R}(v)$ = unit tangent vector of the line $t \mapsto v + t\mathcal{R}(v)$,

$$\begin{aligned}
(d\exp_x)_v(\mathcal{R}(v)) &= \frac{d}{dt}\Big|_{t=0} \exp_x(v + t\mathcal{R}(v)) \\
&= \frac{d}{dt}\Big|_{t=0} \exp_x\left((|v|+t)\frac{v}{|v|}\right) \\
&= \frac{d}{dt}\Big|_{t=0} \gamma(|v|+t) \\
&= \gamma'(|v|)
\end{aligned}$$

$\therefore \ |(d\exp_x)_v(\mathcal{R}(v))| = |\gamma'(|v|)| = |\gamma'(0)| = \left|\frac{v}{|v|}\right| = 1.$

#