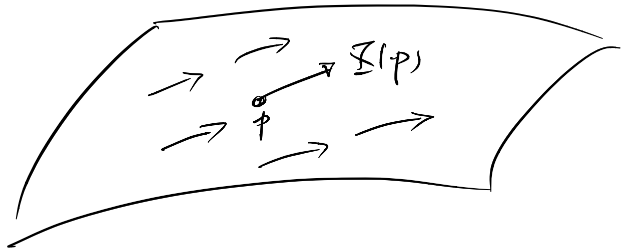
Def: A (smooth) vector field X on a manifold M is a smooth section of the tangent bundle TM,

i.e. $X : M \to TM$ is a smooth map s.t.

$X(p) \in T_pM$

i.e. $\pi \circ X = \mathrm{Id}_M$

$$\begin{tikzcd} TM \arrow[d, "\pi"] \\ M \arrow[u, bend left, "X"] \end{tikzcd}$$

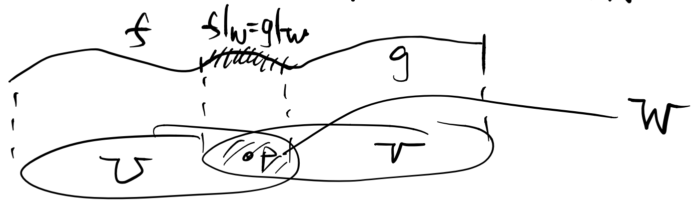


1.5 Tangent vectors as derivations

Let M be a smooth manifold, $p \in M$, consider C^∞ functions defined in a nbd. of p. Then we can define an equivalence relation (Ex!)

$$f : U \to \mathbb{R} \sim g : V \to \mathbb{R} \qquad (p \in U \cap V)$$

$\Leftrightarrow$ $\exists$ nbd $W \subset U \cap V$ of p s.t. $f|_W = g|_W$



Def: The equivalence classes for this relation are the germs of C^∞ functions at p. The space of germs of C^∞ functions at p is denoted by $\mathcal{C}_p^\infty(M)$.

Similarly, we can define $\mathcal{C}_p^0(M)$, $\mathcal{C}_p^k(M)$ & $\mathcal{C}_p^\omega(M)$ germs of continuous, C^k, and (real) analytic functions resp. at p.

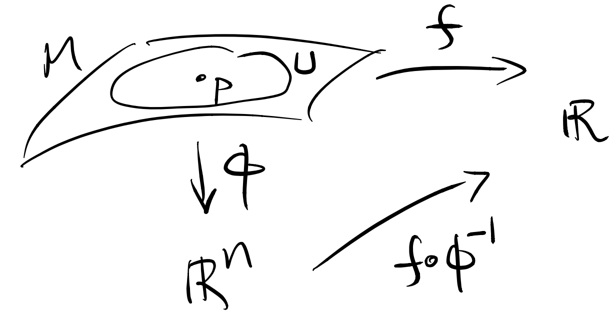
Remark: • Space of functions has linear structure (and a product structure)

$\Rightarrow$ corresponding space of germs is a vector space (with a product structure)

• If M is a C^k manifold $(0 \le k \le \infty)$, then

$$\mathcal{C}_p^k(M) \cong \mathcal{C}_0^k(\mathbb{R}^n) \quad \text{(vector space isomorphism)}$$

Pf (Sketch)



germ of f

$\updownarrow$

germ of $f \circ \phi^{-1}$

Def : A derivation on $\mathcal{C}^k_p(M)$ is a linear map $\delta: \mathcal{C}^k_p(M) \to \mathbb{R}$ such that $\forall f, g \in \mathcal{C}^k_p(M)$

$$\delta(fg) = f(p)\,\delta(g) + g(p)\,\delta(f),$$

where fg = product of the germs f & g

(Ex: How to define fg ?)

Notation: We denote the set of derivations on $\mathcal{C}^k_p(M)$ by $\mathcal{D}^k_p(M)$, or $\mathcal{D}_p(M)$ if k is clear.

Thm: Any derivation of $\mathcal{C}^\infty_0(\mathbb{R}^n)$ can be written as

$$\delta(f) = \sum_{j=1}^{n} \delta(x^j)\,\frac{\partial f}{\partial x^j}(0)$$

is a function representing the germ f.

germ

germ of the coordinate function x^j.

Hence $\dim(\mathcal{D}^\infty_0(\mathbb{R}^n)) = n$.

Pf: $\forall$ germ $f \in \mathcal{C}^\infty_0(\mathbb{R})$, f is represented by a C^∞ function, denoted by f again, in a nbd. of 0.

Then $f(x) - f(0) = \int_0^1 \frac{d}{dt} f(tx)\,dt$

$$= \int_0^1 \sum_{j=1}^{n} \frac{\partial f}{\partial x^j}(tx)\, x^j \, dt$$

$$= \sum_{j=1}^{n} x^j h_j(x)$$

where $h_j(x) = \int_0^1 \frac{\partial f}{\partial x^j}(tx)\,dt$ is C^∞

Then $\delta(f) = \delta(f - f(0))$, since $\delta(\text{const.}) = 0$ (Ex!)

$$= \delta\left(\sum_{j=1}^{n} x^j h_j(x)\right)$$

$$= \sum_{j=1}^{n} \left[\delta(x^j)\, h_j(0) + \cancelto{0}{\left(x^j\big|_0\right)}\, \delta(h_j)\right]$$

$$= \sum_{j=1}^{n} \delta(x^j)\, \frac{\partial f}{\partial x^j}(0) \qquad \#$$

<u>Lemma</u>: $\forall\, \xi \in T_pM$, $L_\xi(f) \overset{\text{def}}{=} (D_p f)(\xi)$, $\forall f \in C_p^\infty(M)$

Then $L_\xi \in \mathcal{D}_p(M)$.

(where $D_p f$ is the differential of a representation of f defined similarly as in Diff. Geom using defn 1 of vector)

(Pf: Ex!)

Thm: $\begin{array}{ccc} T_pM & \longrightarrow & \mathcal{D}_p(M) \\ \psi & & \psi \\ \xi & \longmapsto & L_\xi \end{array}$ is an isomorphism (as vector spaces)

Pf: • $\xi \mapsto L_\xi$ is clearly linear.

• $\operatorname{Ker}(\xi \mapsto L_\xi) = 0$

Pf: Let (U,ϕ) be a chart for M around p with $\phi(p) = 0 \in \mathbb{R}^n$. Then ξ can be represented by $\xi = (U, \phi, u)$ with $u \in T_0\mathbb{R}^n \cong \mathbb{R}^n$.

$\Rightarrow$ $\forall$ C^∞ function f in a nbd. around p

$$L_\xi(f) = D_0(f\circ\phi^{-1})(u) \qquad (\text{Ex!})$$

$$= \sum_{j=1}^n u^j \frac{\partial}{\partial x^j}(f\circ\phi^{-1})(0)$$

where $u = (u^1, \cdots, u^n)$

If $\xi \in \operatorname{Ker}(\xi \mapsto L_\xi)$, then $\forall f$

$$0 = \sum_{j=1}^n u^j \frac{\partial}{\partial x^j}(f\circ\phi^{-1})(0)$$

$$\Rightarrow \quad u^j = 0, \forall j \Rightarrow \xi = 0. \text{※}$$

• Finally $\operatorname{Im}(\xi \mapsto L_\xi) = \mathcal{D}_p(M)$

Pf: $\forall\, \delta \in \mathcal{D}_p(M) \cong \mathcal{D}_0(\mathbb{R}^n)$, By previous thm $\Rightarrow$

$$\delta(f) = \sum_{j=1}^{n} \delta(x^j) \frac{\partial}{\partial x^j} (f \circ \phi^{-1})(0)$$

$$\therefore \quad \delta = L_\xi \text{ for } \xi = \left[\left(U, \phi, \begin{pmatrix} \delta(x^1) \\ \vdots \\ \delta(x^n) \end{pmatrix}\right)\right] \in T_pM.$$

#

<u>Remark</u>: In particular, we have $\underline{\dim T_pM = n}$ with basis corresponds to $\left\{ \frac{\partial}{\partial x^j}\Big|_0 \right\}$ in local coordinates.

$$\left(\text{where } \frac{\partial}{\partial x^j}\Big|_0 \in \mathcal{D}_0(\mathbb{R}^n) \text{ s.t. } \begin{pmatrix} \delta(x^1) \\ \vdots \\ \delta(x^n) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow \begin{matrix} j\text{th} \\ \text{place} \end{matrix} \right)$$

<u>Convention</u>: If (U, ϕ) is a chart around p, and $(x^1, \cdots, x^n)$ are the corresponding coordinate functions

$$x^j : U \xrightarrow{\phi} \mathbb{R}^n \xrightarrow{\pi_j} \mathbb{R}.$$

We denote

$$\left(\frac{\partial}{\partial x^j}\right)_p (f) \stackrel{\text{def}}{=\!=} \frac{\partial}{\partial x^j}(f \circ \phi^{-1})(\phi(p))$$

In this notation

$$L_\xi = \sum_{j=1}^{n} u^j \left(\frac{\partial}{\partial x^j}\right)_p \quad \text{for } \xi = [(U, \phi, u)] \in T_pM.$$

Hence $\left(\frac{\partial}{\partial x^j}\right)_p$ can be regarded as a vector in T_pM

$\Rightarrow$ $\frac{\partial}{\partial x^j}$ is a vector field on $U \subset M$.

If $X^1, \cdots, X^n$ are smooth functions, then

$$X = \sum_{j=1}^{n} X^j \frac{\partial}{\partial x^j} \quad \text{is a vector field on } U$$

corresponding to

$$L_X : C^\infty(U) \to C^\infty(U) \text{ defined by}$$

$$\boxed{(L_X f)(p) = \sum_{j=1}^{n} X^j(p) \left(\frac{\partial f}{\partial x^j}\right)_p .}$$

Thm: The map $X \mapsto L_X$ is an isomorphism between the vector spaces $\Gamma(TM)$ = set of smooth vector field on M and $\mathcal{D}(M)$, where $\mathcal{D}(M)$ = set of derivative δ on M defined by

(i) $\delta : C^\infty(M) \to C^\infty(M)$ linear;

(ii) $\delta(fg) = f\,\delta(g) + g\,\delta(f)$.

(Pf: Omitted

Caution: Analog statement for complex manifold is not true, since cut-off functions are needed.)

Note: If $\delta_1, \delta_2 \in \mathcal{D}(M)$, then $\delta_1 \circ \delta_2 \notin \mathcal{D}(M)$

Lemma: If $\delta_1, \delta_2 \in \mathcal{D}(M)$, then

$$\delta_1 \circ \delta_2 - \delta_2 \circ \delta_1 \in \mathcal{D}(M)$$

(Pf: Ex!)

Def: Let X, Y be vector fields on M. Then $[X,Y]$, the bracket of X & Y, is a vector field corresponding to the derivation $L_X \circ L_Y - L_Y \circ L_X$.

i.e.

$$\boxed{L_{[X,Y]} = L_X \circ L_Y - L_Y \circ L_X}$$

Local formula for $[X, Y]$

If $X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i}$, $Y = \sum_{j=1}^{n} Y^j \frac{\partial}{\partial x^j}$ in some local coordinates

then $$L_X f = \sum_i X^i \frac{\partial f}{\partial x^i}$$

$$\Rightarrow \quad L_Y(L_X f) = \sum_{j,i} \left[Y^j X^i \frac{\partial^2 f}{\partial x^j \partial x^i} + Y^j \frac{\partial X^i}{\partial x^j} \frac{\partial f}{\partial x^i} \right]$$

Similarly for $L_X(L_Y f)$

$$\Rightarrow (L_X L_Y - L_Y L_X) f = \sum_{i,j} \left(X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \right) \frac{\partial f}{\partial x^i}$$

$$\Rightarrow \boxed{\begin{aligned} &[X,Y] = \sum_i Z^i \frac{\partial}{\partial x^i} \\ &\text{where} \quad Z^i = \sum_j \left(X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \right) \end{aligned}}$$

Lemma (Jacobi Identity) For vector fields X, Y, Z,

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

(Pf: Trivial)

1.6 Vector Bundles and Tensors

Def: Let E & B be 2 smooth manifolds and

$\pi : E \to B$ be a smooth map.

(π, E, B) is a vector bundle of rank n if

- π is surjective
- $\exists$ open covering $(U_i)_{i\in\Lambda}$ of B and diffeomorphisms $h_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^n$ such that $\forall\, x \in U_i$, $h_i(\pi^{-1}(x)) = \{x\} \times \mathbb{R}^n$

(hence $\pi^{-1}(x)$ can be regarded as a vector space)

- and such that $\forall\, i, j \in \Lambda$, the diffeomaphism

$$h_i \circ h_j^{-1} : (U_i \cap U_j) \times \mathbb{R}^n \longrightarrow (U_i \cap U_j) \times \mathbb{R}^n$$

are of the form

$$h_i \circ h_j^{-1}(x, v) = (x, g_{ij}(x) v)$$

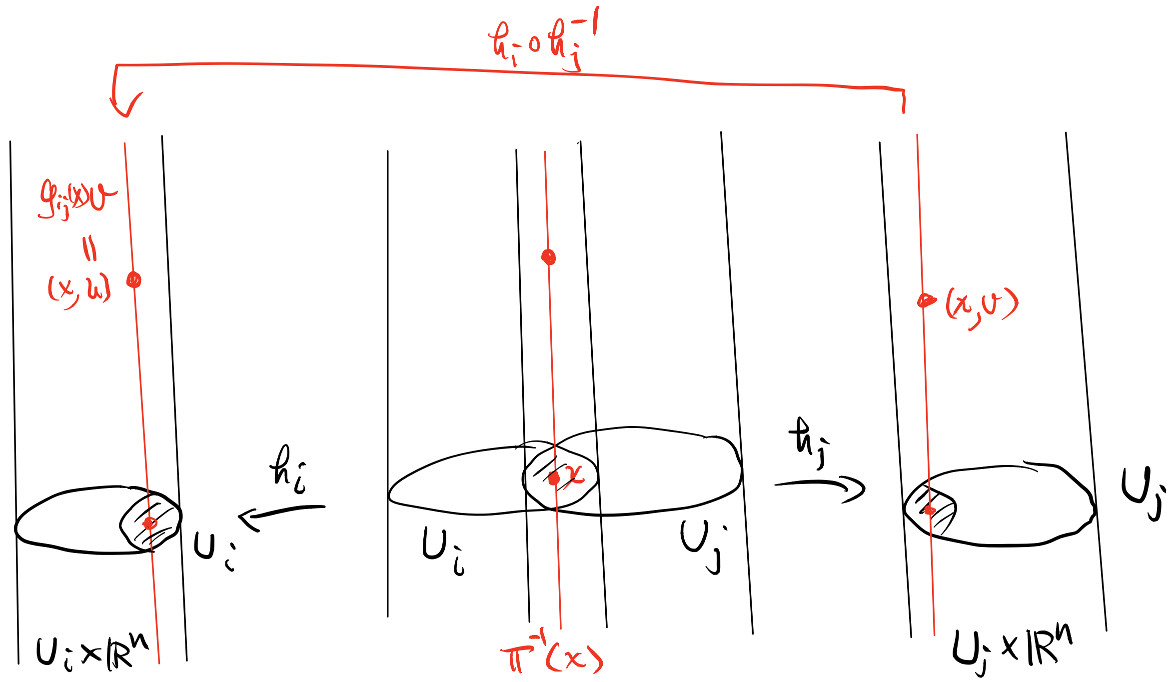
where $g_{ij} : U_i \cap U_j \longrightarrow GL(n, \mathbb{R})$.

<u>Terminology</u>: E = <u>total space</u>

B = <u>base</u>

$\mathbb{R}^n \sim \pi^{-1}(x)$ = <u>fibre</u>

h_i = <u>local trivialization</u>



eg: (Trivial Bundle): $\pi : M \times \mathbb{R}^n \to M$

$$(x, v) \mapsto x$$

eg Tangent bundle of M: $TM = \coprod_{p \in M} T_pM$

(exercise)

Def: (a) A vector bundle of rank n, $\pi : E \to B$, is trivial if $\exists$ diffeomorphism

$$h : E \to B \times \mathbb{R}^n$$

s.t. $h : \pi^{-1}(x) \to \{x\} \times \mathbb{R}^n$ is a vector isomorphism.

(b) A (global) section of the bundle is a smooth map

$s : B \to E$ such that

$\pi \circ s = id_B$

$$\begin{tikzcd} E \arrow[d, "\pi"'] \\ B \arrow[u, bend right, "s"'] \end{tikzcd}$$

eg: vector field $X \in \Gamma(M)$ $(= \Gamma(TM))$ is a section of the tangent bundle TM.

Tensor Product

Def: Let E, F be 2 finite dimensional vector spaces, then $E \otimes F$, the tensor product of E & F, is defined as the vector space, unique up to isomorphism, such that $\forall$ vector space G,

$$L(E \otimes F, G) \overset{\text{isom}}{\cong} L_2(E \times F, G)$$

(linear transformations from $E \otimes F$ to G.) (bilinear maps from $E \times F$ to G)

Remark: $\exists$ a bilinear map $\otimes : E \times F \to E \otimes F$ such that if $\{e_i\}$ = basis for E, and $\{f_j\}$ = basis for F,

then $\{e_i \otimes f_j\}_{i,j}$ is a basis for $E \otimes F$.

Hence for $u = \sum_i a^i e_i \in E$ & $v = \sum_j b^j f_j \in F$

then $$u \otimes v = \sum_{i,j} a^i b^j \, e_i \otimes f_j .$$

Facts: (1) If $E^* = \text{dual of } E = L(E, \mathbb{R})$

$F^* = \text{dual of } F$

then $E^* \otimes F^* \cong L_2(E \times F, \mathbb{R})$

$\cong L(E \otimes F, \mathbb{R}) = (E \otimes F)^*$

$\left(\text{by } \alpha \overset{\psi}{\otimes} \beta \longmapsto \begin{array}{l} \alpha\otimes\beta(u \otimes v) \\ = \alpha(u)\beta(v) \end{array} \right)$

(2) If $\alpha \in L(E, E')$ & $\beta \in L(F, F')$

(E, E', F, F' = finite dim'l vector spaces)

then one can define

$$\alpha \otimes \beta \in L(E \otimes F, E' \otimes F')$$

by $(\alpha \otimes \beta)(u \otimes v) \stackrel{\text{def}}{=\!=} \alpha(u) \otimes \beta(v)$

(3) Given a vector bundle E (with fibers E_x, $x \in M$)

one can define the vector bundle E^*, $\otimes^p E$

(with fibers E_x^* and $\otimes^p E_x$ respectively)

(4) Given 2 vector bundles E, F (with fibers E_x, F_x)

with the <u>same</u> <u>base</u> manifold M, we can define

the vector bundle $E \otimes F$ over M with fiber $E_x \otimes F_x$.

eg: Starting from TM, we can define the <u>cotangent bundle</u>

T^*M of M (with fibers $(T_pM)^*$), and the (p,q)-tensor bundle $\left(\bigotimes^p TM\right)\otimes\left(\bigotimes^q T^*M\right)$ of M.

Def: A (p,q)-tensor (field), or more precisely p times contravariant and q times covariant tensor, on M is a (smooth) section of the bundle $\left(\bigotimes^p TM\right)\otimes\left(\bigotimes^q T^*M\right)$.

Note: For $f: M\to\mathbb{R}$ smooth, we can define

$$df\in\Gamma(T^*M):\quad df(X)=L_X f=Xf \quad \forall X\in\Gamma(M).$$

Then $\{dx^j\}_{j=1}^n$ is a dual (local) basis to $\left\{\frac{\partial}{\partial x^j}\right\}_{j=1}^n$ since $dx^j\left(\frac{\partial}{\partial x^i}\right)=\frac{\partial x^j}{\partial x^i}=\delta_i^j$.

at each point in a coordinate system with coordinate functions $(x^1,\dots,x^n)$.

Then $\left\{\frac{\partial}{\partial x^{j_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{j_p}}\otimes dx^{i_1}\otimes\cdots\otimes dx^{i_q}\right\}$

forms a local basis for $\left(\bigotimes^p TM\right)\otimes\left(\bigotimes^q T^*M\right)$

$\Rightarrow$ in coordinates, a (p,q)-tensor (field) can be written as

$$T = T^{j_1 \cdots j_p}_{i_1 \cdots i_q} \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_p}} \otimes dx^{i_1} \otimes \cdots \otimes dx^{i_q}$$

1.7 Partitions of Unity

Recall that all manifolds in this course are supposed to have the property that "partitions of unity" is always possible. That is,

$\forall \{U_i\}_{i\in\Lambda}$ = open cover of M,

$\exists$ locally finite open cover $\{V_k\}_{k\in\Lambda'}$ and a family $\{\varphi_k\}_{k\in\Lambda'}$ of real smooth functions on M such that

- $\{V_k\}_{k\in\Lambda'}$ is subordinate to $\{U_i\}_{i\in\Lambda}$ (i.e. each $V_k \subset U_i$ for some i)
- $\operatorname{supp} \varphi_k \subset V_k$, $\varphi_k \geq 0$, and $$\sum_{k\in\Lambda'} \varphi_k(x) = 1 \quad \forall x \in M.$$

Here $\{V_k\}_{k\in\Lambda'}$ being locally finite means $\forall x \in M$, $\exists$ open nbd W of x such that $W \cap V_k = \phi$ except finitely many k's.