Prop 4.1 Suppose $S(z)$ is given by (5) in the above definition and $a_1, \cdots, a_n$ & a_∞ are as in the remarks (iii) & (iv).

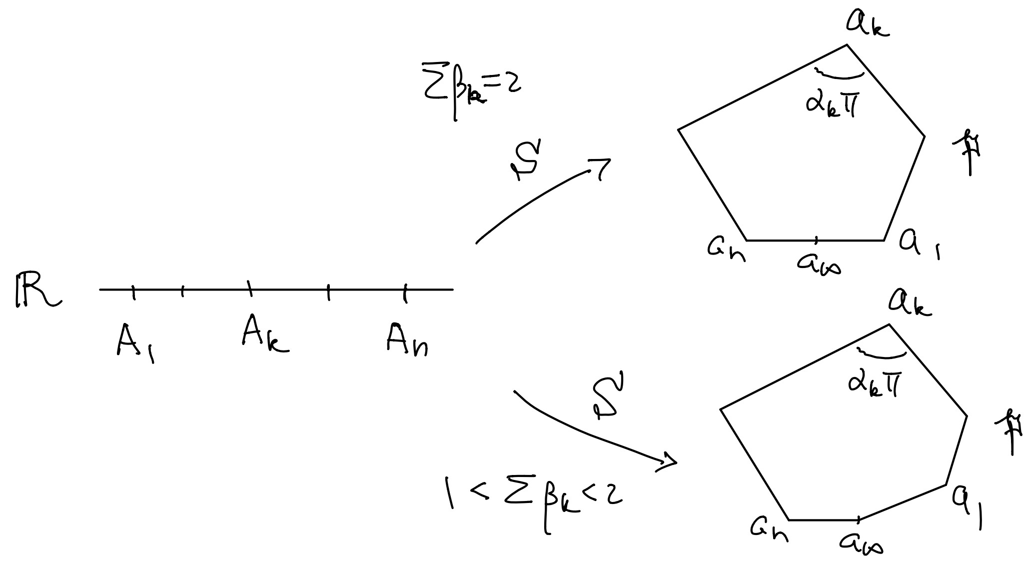
(i) If
- $\sum_{k=1}^{n} \beta_k = 2$, and
- $\mathcal{P}$ denotes the polygon whose vertices are given by $a_1, \cdots, a_n$ (in order)

("polygon" = a closed curve consists of finitely many line segments.)

then
- $a_\infty = S(\infty)$ lies on the segment $[a_n, a_1]$
- $S(\mathbb{R}) = \mathcal{P} \setminus \{a_\infty\}$
- (Interior) angle at $a_k = \alpha_k \pi$, $\alpha_k = 1 - \beta_k$.

(ii) If $1 < \sum_{k=1}^{n} \beta_k < 2$, the similar conclusion holds with
- $\mathcal{P}$ replaced by the polygon of $n+1$ sides with vertices $a_1, a_2, \cdots a_n, a_\infty$ (in order), and
- (Interior) angle at $a_\infty = \alpha_\infty \pi$,

$$\alpha_\infty = 1 - \beta_\infty \quad \& \quad \beta_\infty = 2 - \sum_{k=1}^{n} \beta_k.$$



Pf Case (i) $\sum_{k=1}^{n} \beta_k = 2$

If $A_k < x < A_{k+1}$, $k = 1, \cdots, n-1$.

Then $$S'(x) = \frac{1}{\left[(x-A_1)^{\beta_1} \cdots (x-A_k)^{\beta_k}\right]\left[(x-A_{k+1})^{\beta_{k+1}} \cdots (x-A_n)^{\beta_n}\right]}$$

By the choice of branch of each $x - A_j$ in Remark (ii),

$$\arg (x-A_j)^{\beta_j} = \begin{cases} 0 & \text{for } j \leq k \\ \pi \beta_j & \text{for } j > k \end{cases}$$

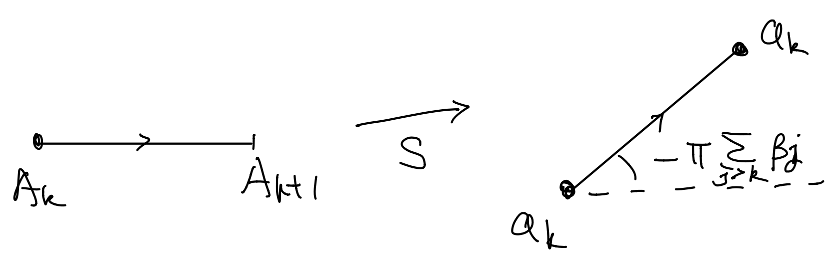
$\therefore$ $\arg S'(x) = -\pi \sum_{j>k} \beta_j$

which is a constant for $x \in (A_k, A_{k+1})$.

$\Rightarrow$ $S[A_k, A_{k+1}]$ is a straight line segment that makes an angle of $-\pi \sum_{j>k} \beta_j$ with the x-axis.

Notice that $S(x) = S(A_k) + \int_{A_k}^{x} S'(y)\,dy$ $\forall x \in (A_k, A_{k+1})$.

$S(x)$ varies from end point $a_k = S(A_k)$ to end point $a_{k+1} = S(A_{k+1})$ as x varies from A_k to A_{k+1}.



Similarly $$\arg S'(x) = \begin{cases} 0 & \text{if } x > A_n \quad (\text{i.e. } S'(x) > 0) \\ -\pi \sum_{k=1}^{n} \beta_k = -2\pi, & \text{if } x < A_1 \end{cases}$$

And • $S(x)$ varies from $a_n = S(A_n)$ to $a_\infty = S(A_\infty)$

as x varies from A_n to ∞.

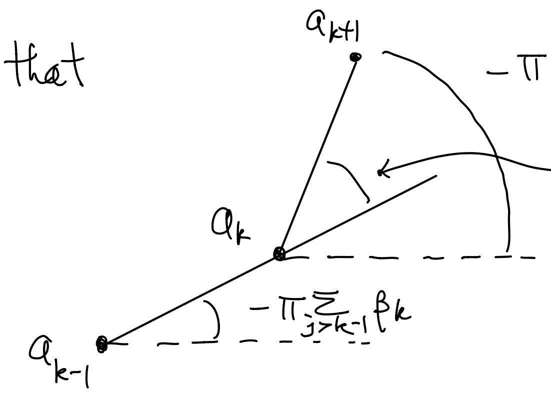
• $S(x)$ varies from a_∞ to $a_1 = S(A_1)$

as x varies from $-\infty$ to A_1

This shows that $a_\infty \in [a_1, a_n]$ (angles with x-axis $= 0$ & -2π)

This proves $S(\mathbb{R}) = \mathcal{P} \setminus \{a_\infty\}$.

Note that

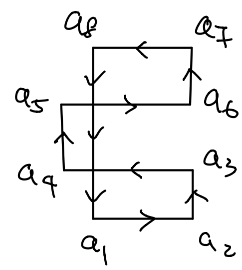


$$\pi\beta_k = \left(-\pi \sum_{j>k} \beta_k\right) - \left(-\pi \sum_{j>k-1} \beta_k\right)$$

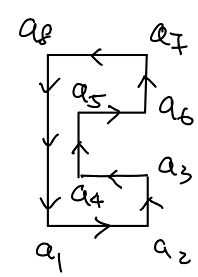
$\therefore$ Interior angle at $a_k = \pi - (\pi\beta_k) = \alpha_k \pi$.

Case (ii) $1 < \sum_{k=1}^{n} \beta_k < 2$ is similar (Ex!) ##

Notes: (i) For an arbitrary choice of n; $A_1, \cdots, A_n$; $\beta_1, \cdots, \beta_n$, the "polygon" $\mathcal{P}$ in Prop 4.1 may not be simple. The following could happen:



(compares



)

(ii) Even $\mathcal{P} = \partial P$, P simply-connected region, Prop 4.1 hasn't shown that $S : \mathbb{H} \to P$ is conformal. (See subsection 4.4 below)

4.3 Boundary Behavior

Let P = polygonal region with boundary $\mathcal{P}$ (polygon)

Then P is bounded, simply-connected open & connected.

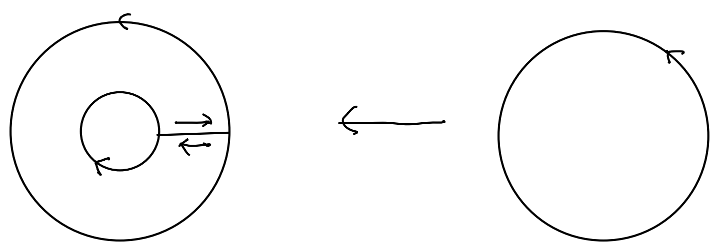
Thm 4.2 If $F: \mathbb{D} \to P$ is a conformal map, then F extends to a continuous bijection from the closure $\overline{\mathbb{D}}$ to the closure $\overline{P}$.

In particular $F|_{\partial \mathbb{D}} : \partial \mathbb{D} \to \mathcal{P}$ cts. & bijective.

Pf: Omitted (as it's more of a real (geometric) analysis argument, and techincal.)

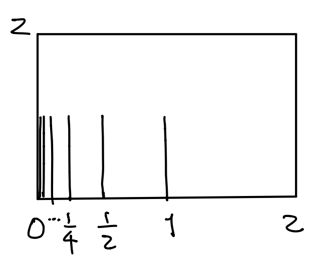
Remark: Thm 4.2 is not true for general proper simply-connected regions. It is true $\Leftrightarrow$ $\partial\Omega$ is a Jordan curve.

Eg1: Ω =



Boundary map cannot be injective. (Proof Omitted)

Eg2: Ω =



$(0,2)\times(0,2) \setminus \bigcup_{n=1}^{\infty} \{ \frac{1}{n} + iy : 0 < y \leq 1 \}$

is simply-connected proper regions. But $F: \mathbb{D} \to \Omega$ cannot be extended continuously to $\partial \mathbb{D}$. (Proof omitted)

4.4 The Mapping Formula

Let
- P = bounded polygon region
- $\mathcal{P}$ = boundary polygon of P
- $a_1, a_2, \cdots, a_n$ ordered vertices of $\mathcal{P}$ ($n \geq 3$).
- $\pi \alpha_k$ = interior angle at a_k.
- $\pi \beta_k$ = exterior angle at a_k, i.e. $\beta_k = 1 - \alpha_k$

Then $\sum_{k=1}^{n} \beta_k = 2$ (Elementary Euclidean Geometry)

Let $F : \mathbb{H} \to P$ be <u>conformal</u>

- Existence is guaranteed by Riemann mapping thm:

$$\begin{tikzcd} \mathbb{H} \arrow[r] \arrow[rr, bend left, "F"] & \mathbb{D} \arrow[r] & P \end{tikzcd}$$

$$z \longmapsto w = \frac{i-z}{i+z} \xmapsto[\text{Riemann map}]{} G(w) = F(z)$$

- Since G extends continuously to $\overline{\mathbb{D}}$ by Thm 4.2 and $z \mapsto w = \frac{i-z}{i+z}$ clearly extends continuously to the boundary x-axis,
  the conformal map $F : \mathbb{H} \to P$ extends continuously to $\overline{\mathbb{H}}$.

- May assume $A_k = F^{-1}(a_k) \in \mathbb{R}$ (i.e. no vertex of $\mathcal{P} \Leftrightarrow \infty$)

- F continuous & bijective
  $\Rightarrow \quad A_1 < \cdots < A_n$ (by relabeling a_k if needed)

- Then $\quad [A_k, A_{k+1}] \ (1 \le k \le n-1) \xrightarrow{F} [a_k, a_{k+1}]$

  $(-\infty, A_1] \cup [A_n, \infty) \xrightarrow{F} [a_n, a_1]$

Thm 4.6 Let $F : \mathbb{H} \to P$ conformal, s.t. $F(\infty)$ is not a vertex of P.
S = Schwarz-Christoffel integral in subsection 4.2
with A_k & β_k as above
Then $\exists$ (cpx) constants C_1 and C_2 such that

$$F(z) = C_1 S(z) + C_2 . \qquad (C_1 \neq 0)$$

Idea of proof: If $F = C_1 S + C_2$,
then
$$F'(z) = \frac{C_1}{(z-A_1)^{\beta_1} \cdots (z-A_n)^{\beta_n}}$$

$$\Rightarrow \quad \log F'(z) = \log C_1 - \sum_{k=1}^{n} \beta_k \log (z - A_k) \qquad \text{(whenever defined)}$$

$$\Rightarrow \quad \frac{F''(z)}{F'(z)} + \sum_{k=1}^{n} \frac{\beta_k}{z - A_k} = 0 .$$

Hence we need to study : $\begin{cases} \text{(i) behavior of } F \text{ at } A_k \\ \text{(ii) behavior of } F \text{ at } \infty . \end{cases}$

Hope that (i) gives the correct singularities at A_k &
(ii) to conclude it is the constant zero.