$$\boxed{\Gamma : \mathrm{Aut}(\mathbb{D}) \to \mathrm{Aut}(\mathbb{H}) \text{ is a group isomorphism}}$$

Pf: We've seen that Γ is bijection, it remains to check that Γ is a group homomorphism: $\forall\, \varphi_1, \varphi_2 \in \mathrm{Aut}(\mathbb{D})$,

$$\begin{aligned}\Gamma(\varphi_1 \circ \varphi_2) &= F^{-1} \circ (\varphi_1 \circ \varphi_2) \circ F \\ &= F^{-1} \circ \varphi_1 \circ F \circ F^{-1} \circ \varphi_2 \circ F \\ &= \Gamma(\varphi_1) \circ \Gamma(\varphi_2) \quad ※\end{aligned}$$

Remark: This fact can be generalized to any conformally equivalent open sets U and V.

Explicit description of $\mathrm{Aut}(\mathbb{H})$

$$\boxed{\begin{aligned}&\text{Thm 2.4}: \ f \in \mathrm{Aut}(\mathbb{H}) \Leftrightarrow \\ &\qquad f(z) = \frac{az+b}{cz+d} \quad \text{for some } a,b,c,d \in \mathbb{R} \\ &\qquad\qquad\qquad\qquad \& \ ad-bc = 1. \ (>0 \text{ suff.})\end{aligned}}$$

Remarks: (i) $a,b,c,d \in \mathbb{R}$, not just $\mathbb{C}$.

(ii) any $f \in \mathrm{Aut}(\mathbb{H})$ is a fractional linear transformation

(iii) any $f \in \mathrm{Aut}(\mathbb{D})$ is a fractional linear transformation

Pf of Thm 2.4

($\Leftarrow$) $ad-bc = 1 \Rightarrow (a,b), (c,d)$ are linearly independent

(in particular c, d can't be 0 simultaneously)

$\therefore \ f(z) = \dfrac{az+b}{cz+d}$ is well-defined (and non-constant)

$c, d \in \mathbb{R} \Rightarrow f$ is holo in $\mathbb{H}$.

Now $f(x+iy) = \dfrac{a(x+iy)+b}{c(x+iy)+d} = \dfrac{(ax+b)+iay}{(cx+d)+icy}$

$$= \frac{[(ax+b)+iay][(cx+d)-icy]}{(cx+d)^2+c^2y^2}$$

$$\Rightarrow \operatorname{Im} f(z) = \frac{ay(cx+d) - cy(ax+b)}{(cx+d)^2+c^2y^2} = \frac{(ad-bc)y}{|cz+d|^2}$$

$$= \frac{y}{|cz+d|^2} > 0 \quad \forall y > 0$$

$\therefore f : \mathbb{H} \to \mathbb{H}$.

Observe that $g(z) = \dfrac{dz-b}{-cz+a}$ has the same form

with coefficients satisfying $d \cdot a - (-b)(-c) = 1$.

$\therefore$ g is well-defined, holo in $\mathbb{H}$ and

$g : \mathbb{H} \to \mathbb{H}$.

Straight faward calculation:

$$f \circ g(z) = \frac{a\left(\frac{dz-b}{-cz+a}\right)+b}{c\left(\frac{dz-b}{-cz+a}\right)+d} = \frac{a(dz-b)+b(-cz+a)}{c(dz-b)+d(-cz+a)}$$

$$= \frac{(ad-bc)z}{(ad-bc)} = z$$

Similarly $g \circ f(z) = z$.

$\therefore$ $g = f^{-1}$ and hence $f \in \operatorname{Aut}(\mathbb{H})$.

($\Rightarrow$) If $f \in \operatorname{Aut}(\mathbb{H})$, then $\beta \overset{\text{denote}}{=} f^{-1}(i) \in \mathbb{H}$.

If $\beta = u + iv$, $u, v \in \mathbb{R}$, $v > 0$.

Then $\psi(z) = \dfrac{z-u}{v} = \dfrac{\frac{1}{\sqrt{v}} z + \left(-\frac{u}{\sqrt{v}}\right)}{0 \cdot z + \sqrt{v}} \in \mathrm{Aut}(\mathbb{H})$

as $\frac{1}{\sqrt{v}} \cdot \sqrt{v} - \left(-\frac{u}{\sqrt{v}}\right) \cdot 0 = 1$.

And $\psi(\beta) = \dfrac{(u+iv)-u}{v} = i$ & hence $\psi^{-1}(i) = \beta$

Consider $g = f \circ \psi^{-1} \in \mathrm{Aut}(\mathbb{H})$.

Then $g(i) = f \circ \psi^{-1}(i) = f(\beta) = i$.

$\Rightarrow$ $\Gamma^{-1}(g) = F \circ g \circ F^{-1} \in \mathrm{Aut}(\mathbb{D})$, where $F(z) = \frac{i-z}{i+z}$,

satisfies

$$\Gamma^{-1}(g)(0) = F \circ g \circ F^{-1}(0) = F \circ g(i) = F(i) = 0$$

Schwarz lemma $\Rightarrow$ $\Gamma^{-1}(g)(z) = e^{i2\theta} z$ for some $\theta \in \mathbb{R}$

$$\Rightarrow \quad g(z) = F^{-1} \circ \Gamma^{-1}(g) \circ F(z) = F^{-1}\left(e^{i2\theta} F(z)\right)$$

$$= i \, \frac{1 - e^{i2\theta}\left(\frac{i-z}{i+z}\right)}{1 + e^{i2\theta}\left(\frac{i-z}{i+z}\right)}$$

$$= i \, \frac{(1+e^{i2\theta})z + i(1-e^{i2\theta})}{(1-e^{i2\theta})z + i(1+e^{i2\theta})}$$

$$= i \, \frac{(e^{i\theta}+e^{-i\theta})z - i(e^{i\theta}-e^{-i\theta})}{-(e^{i\theta}-e^{-i\theta})z + i(e^{i\theta}+e^{-i\theta})}$$

$$\Rightarrow \quad f \circ \psi^{-1}(z) = \frac{\cos\theta \cdot z + \sin\theta}{-\sin\theta \cdot z + \cos\theta}$$

$$\therefore \quad f(z) = \frac{\cos\theta \cdot \left(\frac{z-u}{v}\right) + \sin\theta}{-\sin\theta \cdot \left(\frac{z-u}{v}\right) + \cos\theta}$$

$$= \frac{\frac{\cos\theta}{\sqrt{v}} \cdot z + \frac{(-u\cos\theta + v\sin\theta)}{\sqrt{v}}}{-\frac{\sin\theta}{\sqrt{v}} \cdot z + \frac{(u\sin\theta + v\cos\theta)}{\sqrt{v}}}$$

Clearly coefficients are real and

$$\frac{\cos\theta}{\sqrt{v}} \frac{(u\sin\theta + v\cos\theta)}{\sqrt{v}} - \left(\frac{-\sin\theta}{\sqrt{v}}\right)\frac{(-u\cos\theta + v\sin\theta)}{\sqrt{v}}$$

$$= \cos^2\theta + \sin^2\theta = 1$$

$\therefore$ f is of the required form. ※

Remark: The proof in the Textbook uses the following relationship between fractional linear transformations and 2×2 matrixes.

$$f_M(z) = \frac{az+b}{cz+d} \longleftrightarrow M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Note that (i) $f_I = \mathrm{Id}$, where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(ii) $f_{M_1} \circ f_{M_2} = f_{M_1 M_2}$

Pf: $$\frac{a_1\left(\frac{a_2 z + b_2}{c_2 z + d_2}\right) + b_1}{c_1\left(\frac{a_2 z + b_2}{c_2 z + d_2}\right) + d_1} = \frac{a_1(a_2 z + b_2) + b_1(c_2 z + d_2)}{c_1(a_2 z + b_2) + d_1(c_2 z + d_2)}$$

$$= \frac{(a_1 a_2 + b_1 c_2) z + (a_1 b_2 + b_1 d_2)}{(c_1 a_2 + d_1 c_2) z + (c_1 b_2 + d_1 d_2)}$$

(provided $(c_1, d_1) \neq 0$ & $(c_2, d_2) \neq 0$)

(iii) By (i) & (ii), $(f_M)^{-1}$ exists $\Longleftrightarrow$ M^{-1} exists

and $(f_M)^{-1} = f_{(M^{-1})}$

(iv) However, $f_{(-M)} = f_M$. (In fact $f_{(kM)} = f_M$, $\forall k \in \mathbb{C}\setminus\{0\}$)

For the purpose of proving Thm 2.4, the Textbook considered

$$SL_2(\mathbb{R}) = \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{R} \ \& \ \det(M) = ad-bc = 1 \right\}$$

((real) <u>special linear group</u> (of degree 2))

and Thm 2.4 can be written as

$$\underset{\left(\begin{matrix}\cong \\ \mathrm{Aut}(\mathbb{D})\end{matrix}\right)}{\mathrm{Aut}(\mathbb{H})} \overset{\text{group iso}}{\cong} SL_2(\mathbb{R}) / \pm I \overset{\text{def}}{=\!=} PSL_2(\mathbb{R}) \left(\begin{matrix} \text{(real) projective} \\ \text{special linear group} \\ \text{(of degree 2)}. \end{matrix} \right)$$

§3 The Riemann Mapping Theorem

3.1 Necessary Conditions and Statement of the Theorem

The Problem: determine conditions on an (nonempty) open set Ω that guarantee the existence of conformal map $F: \Omega \to \mathbb{D}$.

(Then for Ω satisfying these conditions, Dirichlet problem in Ω is solvable.)

Necessary conditions

(1) If $F: \Omega \to \mathbb{D}$ conformal, then

$$\sup_{z \in \Omega} |F(z)| = 1$$

Therefore $\Omega \neq \mathbb{C}$, otherwise Liouville's Thm

$\Rightarrow$ $F(z) \equiv$ const. which cannot be conformal.

For convenience, let we call a non-empty set Ω proper if $\Omega \neq \mathbb{C}$.

(2) If $F: \Omega \to \mathbb{D}$ conformal, then $F: \Omega \to \mathbb{D}$ is a homeomorphism and hence Ω and $\mathbb{D}$ are topological equivalent. In particular, Ω must be simply-connected region.

(open and connected in $\mathbb{C}$)

Thm 3.1 (Riemann Mapping Theorem)

Suppose region Ω is proper and simply-connected. Then $\forall z_0 \in \Omega$, $\exists$ a unique conformal map

$$F : \Omega \to \mathbb{D} \quad \text{such that } F(z_0) = 0 \text{ and } F'(z_0) > 0.$$

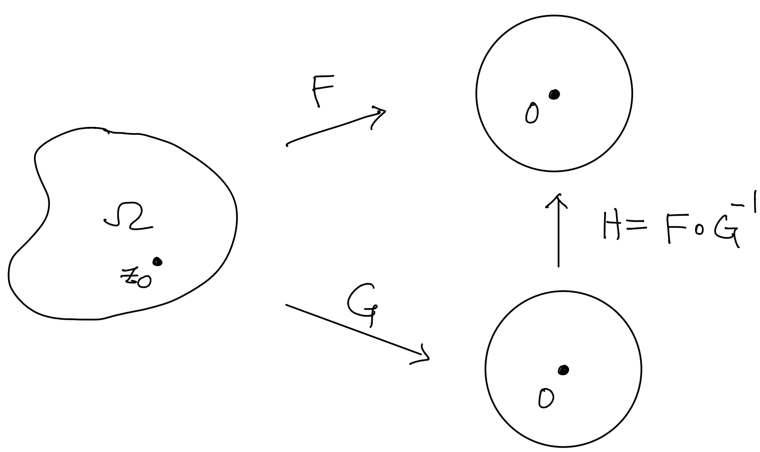
This means $F'(z_0) \in \mathbb{R}$ and $F'(z_0) > 0$.

Cor 3.2 Any two proper simply connected regions in $\mathbb{C}$ are conformally equivalent.

Remark: Hence simply connected regions in $\mathbb{C}$ fall into only 2 conformal equivalent classes: $\mathbb{C}$ or $\mathbb{D}$.

Proof of uniqueness of Thm 3.1.

Suppose that $F : \Omega \to \mathbb{D}$, $G : \Omega \to \mathbb{D}$ are conformal and satisfying
$$\begin{cases} F(z_0) = G(z_0) = 0 \\ F'(z_0) > 0, \quad G'(z_0) > 0. \end{cases}$$



Then $\quad H: \mathbb{D} \to \mathbb{D}$ conformal,

and $\quad H(0) = F \circ G^{-1}(0) = F(z_0) = 0$

$\therefore \quad H \in \mathrm{Aut}_0(\mathbb{D})$.

By Schwarz Lemma (more precisely Cor 2.3),

$$H(z) = e^{i\theta} z \qquad \text{for some } \theta \in \mathbb{R}.$$

$$\Rightarrow \quad e^{i\theta} = H'(0) = F'(G^{-1}(0)) \frac{1}{G'(G^{-1}(0))} = \underbrace{\frac{F'(z_0)}{G'(z_0)} > 0}_{\text{real and positive}}$$

$\therefore \quad e^{i\theta} = 1$

And hence $\quad F \circ G^{-1}(z) = z \iff F \equiv G$. ※

Existense part is much harder and will be handled in the next two subsections.

3.2 Montel's Theorem

Def: Let $\Omega \subset \mathbb{C}$ be open. A family $\mathcal{F}$ of holomorphic functions on Ω is said to be normal if every sequence in $\mathcal{F}$ has a subsequence that converges uniformly on every compact subset of Ω

($\mathcal{F}$ is called precompact if one can make the convergence as a convergence of a metric (Ω, d). See MATH3060.)

Def: Let $\Omega \subset \mathbb{C}$ be open. A family $\mathcal{F}$ of holomorphic functions on Ω is said to be

(1) <u>uniformly bounded on compact subsets</u> of Ω if $\forall$ compact set $K \subset \Omega$, $\exists B > 0$ such that
$$|f(z)| \le B, \ \forall z \in K \text{ and } f \in \mathcal{F}.$$

(2) <u>equicontinuous</u> on a compact set K if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that whenever $z, w \in K$ with $|z - w| < \delta$, then
$$|f(z) - f(w)| < \varepsilon, \ \forall f \in \mathcal{F}.$$

(Ex: review MATH3060 on the related properties)

In metric space setting of family of continuous functions, the properties (1) and (2) are independent. However, for family of holomorphic functions, $(1) \Rightarrow (2)$, thanks to the Cauchy Integral Formula:

Thm 3.3 Suppose $\mathcal{F}$ is a family of holomorphic functions on Ω that is <u>uniformly bounded on compact subsets</u> of Ω. Then

(i) $\mathcal{F}$ is <u>equicontinuous on every compact subset</u> of Ω.

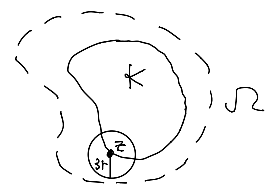
(ii) $\mathcal{F}$ is a <u>normal</u> family.

Pf of (i)

Let $K \subset \Omega$ be compact.

Then $\exists\, r > 0$ such that $\forall z \in K$, $D_{3r}(z) \subset \Omega$

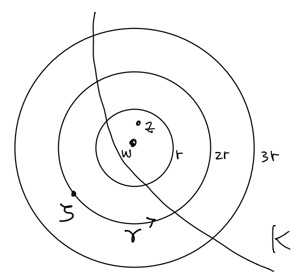
$\left(0 < r < \frac{1}{3}\operatorname{dist}(K, \partial\Omega)\right)$



If $z, w \in K$ and $|z-w| < r$.

Let $\gamma = \partial D_{2r}(w)$

Then Cauchy's integral formula



$$\Rightarrow \quad f(z) - f(w) = \frac{1}{2\pi i}\int_\gamma f(\zeta)\left[\frac{1}{\zeta - z} - \frac{1}{\zeta - w}\right] d\zeta$$

$$\Rightarrow \quad |f(z) - f(w)| \le \frac{1}{2\pi}\int_\gamma |f(\zeta)| \left|\frac{1}{\zeta - z} - \frac{1}{\zeta - w}\right| |d\zeta|$$

$$= \frac{1}{2\pi}\int_\gamma |f(\zeta)| \frac{|z-w|}{|\zeta - z||\zeta - w|} |d\zeta|$$

$$\le \frac{1}{2\pi}\int_\gamma |f(\zeta)| \frac{|z-w|}{r^2} |d\zeta|$$

By assumption, $\exists\, B > 0$ s.t. $|f(z)| \le B$, $\forall z \in \Omega$ & $f \in \mathcal{F}$,

we have $|f(z) - f(w)| \le \frac{B|z-w|}{r^2} \cdot \frac{1}{2\pi} \cdot 2\pi(2r) = \frac{2B}{r}|z-w|$

$\forall z, w \in K$, $|z-w| < r$ & $\forall f \in \mathcal{F}$.

This implies $\mathcal{F}$ is equicontinuous (Ex!) ※

To prove (ii), we need the following

Lemma 3.4 Any open set Ω $(\subset \mathbb{C})$ has a compact exhaustion

Recall:

A compact exhaustion (simple called exhaustion in the Textbook) of Ω is a sequence $\{K_\ell\}_{\ell=1}^{\infty}$ of compact subsets of Ω such that

(i) $K_\ell \subset \operatorname{int}(K_{\ell+1})$ $\forall\ \ell=1,2,3,\cdots$

(ii) $\forall$ compact subset K of Ω, $\exists\ K_\ell$ such that $K \subset K_\ell$.

In particular, $\Omega = \bigcup_{\ell=1}^{\infty} K_\ell$.

Pf of Lemma 3.4:

If Ω is bounded, $K_\ell = \{z \in \Omega : \operatorname{dist}(z, \partial\Omega) \geq \frac{1}{\ell}\}$ is the required compact exhaustion.

If Ω is unbounded,

$$K_\ell = \{z \in \Omega : \operatorname{dist}(z, \partial\Omega) \geq \tfrac{1}{\ell} \text{ and } |z| \leq \ell\}$$

is the required compact exhaustion.

(Ex: give the details) ※

Pf of (ii) (of Thm 3.3)

Let $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$ be a sequence.

Let $K \subset \Omega$ be compact.

Then by (i), $\{f_n\}_{n=1}^{\infty}$ is uniformly bounded and equicontinuous on K.

Arzela-Ascoli Theorem (on the metric space $C(K, d_\infty)$)
(review MATH3060)

$\Rightarrow$ $\exists$ subsequence of $\{f_n\}$ converges uniformly on K.

Let $\{K_\ell\}_{\ell=1}^{\infty}$ be a compact exhaustion of Ω.

Then $\{f_n\}$ has a convergent subsequence $\{g_{n,1}\}$ on K_1
(in uniform metric)

Applying the same argument,

$\{g_{n,1}\}$ has a convergent subseq $\{g_{n,2}\}$ on $K_2 \supset K_1$
(in uniform metric)

And so on, we have subseq. $\{g_{n,\ell}\}$ of $\{f_n\}$ satisfying

(i) $\{g_{n,\ell}\}$ converges uniformly on $K_\ell \supset \cdots \supset K_1$

(ii) $\{g_{n,\ell+1}\}$ is a subseq. of $\{g_{n,\ell}\}$.

Then the seq. $\{g_{n,n}\}$ is a subsequence of $\{f_n\}$ that converges uniformly on K_ℓ, $\forall \ell = 1, 2, \cdots$

Since $\{K_\ell\}$ is a compact exhaustion of Ω,
$\{g_{n,n}\}$ converges uniformly on any compact $K \subset \Omega$ (as $K \subset K_\ell$ for some ℓ). This proves $\mathcal{F}$ is normal.

$\#$