Ch 8 Conformal Mappings

§1 Conformal Equivalence and Examples

Def • A bijective holomorphic function $f: U \to V$ (U, V open in $\mathbb{C}$) is called a conformal map or biholomorphism.

• In this case, U and V are said to be conformally equivalent or simply biholomorphic.

Prop 1.1 • If $f: U \to V$ is holomorphic and injective, then

$$f'(z) \neq 0 \quad \forall z \in U.$$

• In particular, $f^{-1}: f(U) \to U$ is holomorphic
($\Rightarrow$ inverse of conformal map is also holomorphic and hence conformal)

Remarks: • Prop 1.1 $\Rightarrow$

U & V are conformally equivalent
$\Leftrightarrow$ $\exists$ holo. $f: U \to V$ and $g: V \to U$ s.t.

$$g(f(z)) = z \quad \forall z \in U \quad \&$$
$$f(g(w)) = w \quad \forall w \in V.$$

• Some authors call a holomorphic map $f: U \to V$ conformal if $f'(z) \neq 0$, $\forall z \in U$, not necessary bijective (globally)
(In this course, we'll follow Textbook's convention.)

Pf of Prop 1.1

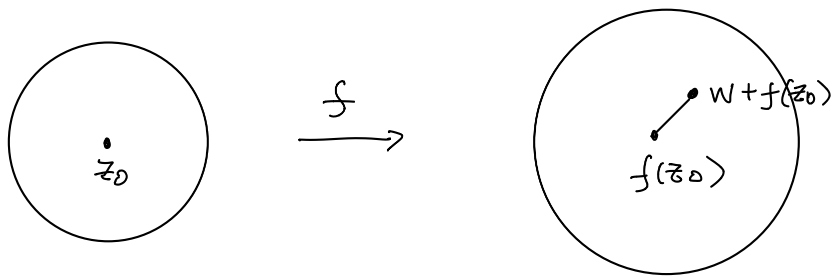
Suppose on the contrary that $f'(z_0)=0$ for some $z_0 \in U$.

Then $f(z)-f(z_0) = a(z-z_0)^k + G(z)$ near z_0

where $a \neq 0$, $k \geq 2$ and

G vanishing to order $k+1$ at z_0.

$\left(\text{i.e. } \dfrac{|G(z)|}{|z-z_0|^{k+1}} \leq C \text{ near } z_0 \right)$



Consider $w \neq 0$ with $|w|$ sufficiently small

Then

$$f(z)-(f(z_0)+w) = [a(z-z_0)^k - w] + G(z)$$
$$= F(z) + G(z)$$

where $F(z) = a(z-z_0)^k - w$.

Then
- $|F(z)| > |G(z)|$ on a sufficiently small circle centered at z_0
- $k \geq 2 \Rightarrow F(z)$ has at least 2 zeros inside that circle.

Rouché's Thm $\Rightarrow$

$f(z)-(f(z_0)+w)$ has at least 2 zeros there two.

Since f' is holo & hence z_0 is an isolated zero.

We may assume, by choosing a smaller circle, that

$f'(z) \neq 0 \quad \forall z$ insides the circle except $z = z_0$.

$\Rightarrow$ the zeros of $f(z) - (f(z_0) + w)$ are distinct

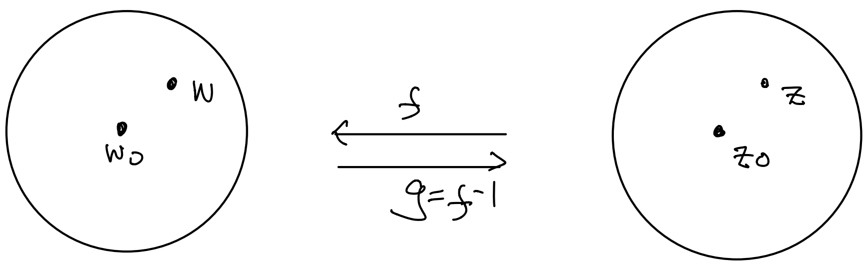
$\therefore$ f is not injective near z_0.

This proves the 1st statement.

For the 2nd statement, let $g = f^{-1} : f(U) \to U$

(Open mapping theorem (Thm 4.4, Ch 3) $\Rightarrow$ g is continuous)

Suppose that $w_0 \in f(U)$ and w close to w_0, but $w \neq w_0$.



Then $\exists\, z$ & $z_0 \in U$ s.t. $w = f(z)$ & $w_0 = f(z_0)$.

Hence $$\frac{g(w) - g(w_0)}{w - w_0} = \frac{1}{\left(\frac{f(z) - f(z_0)}{z - z_0}\right)}$$

Since $f'(z_0) \neq 0$, we have

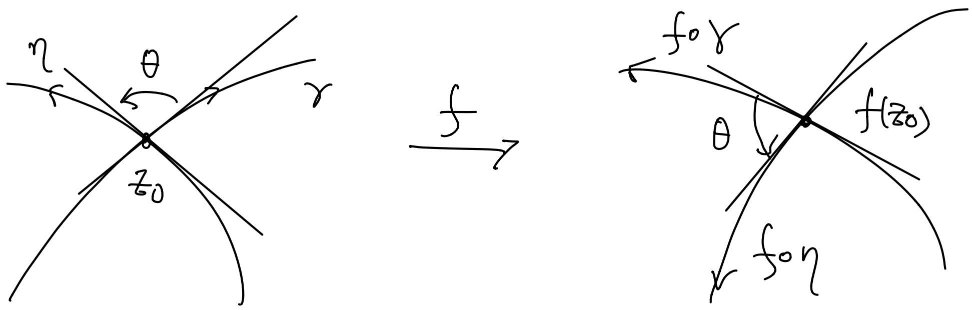
$$\lim_{w \to w_0} \frac{g(w) - g(w_0)}{w - w_0} = \frac{1}{\lim_{z \to z_0}\left(\frac{f(z) - f(z_0)}{z - z_0}\right)} = \frac{1}{f'(z_0)} \text{ exists}$$

$\therefore$ g is holo. and $g'(w_0) = \dfrac{1}{f'(g(w_0))}$ ※

<u>Remark</u>: $\boxed{\text{If } f : U \to \mathbb{C},\ z_0 \in U, \text{ and } f'(z_0) \neq 0. \text{ Then } f \text{ } \underline{\text{preserves angles}} \text{ at } z_0.}$

The precise formulation is:

Let γ & η be two (smooth oriented) curves intersecting at z_0, then the angle from the curve $f\circ\gamma$ to the curve $f\circ\eta$ at $f(z_0)$ <u>equals</u> the angle from the curve γ to the curve η at z_0.



(Problem 2 on page 255 of the Textbook.)

Hence

$$\boxed{\text{conformal maps preserve angles}}$$

1.1 <u>The disc and upper half-plane</u>

<u>Notations</u>:
- (unit) Disc $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$
- upper half-plane $\mathbb{H} = \{ z \in \mathbb{C} : \operatorname{Im}(z) > 0 \}$

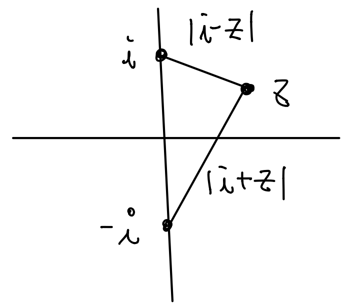
<u>Thm 1.2</u> : The map $F : \mathbb{H} \to \mathbb{D}$

$$z \mapsto \frac{i-z}{i+z}$$ is a conformal map

with inverse $G = F^{-1} : \mathbb{D} \to \mathbb{H}$

$$w \mapsto i\,\frac{1-w}{1+w}.$$

<u>Pf</u> : Clearly $\begin{cases} z \in \mathbb{H} \Rightarrow i+z \neq 0 \Rightarrow F \text{ is holo.} \\ w \in \mathbb{D} \Rightarrow 1+w \neq 0 \Rightarrow G \text{ is holo.} \end{cases}$



Then • $|F(z)| = \left| \frac{i-z}{i+z} \right| < 1$

$\Rightarrow \quad F(\mathbb{H}) \subset \mathbb{D}$,

• And for $w = u + iv \in \mathbb{D}$,

$$\operatorname{Im}(G(w)) = \operatorname{Im}\left(i\,\frac{1-u-iv}{1+u+iv} \right)$$

$$= \frac{1-u^2-v^2}{(1+u)^2+v^2} > 0$$

$\therefore \quad G(\mathbb{D}) \subset \mathbb{H}$.

Finally $$F(G(w)) = \frac{i - i\,\frac{1-w}{1+w}}{i + i\,\frac{1-w}{1+w}} = w$$

& $$G(F(z)) = i \cdot \frac{1 - \frac{i-z}{i+z}}{1 + \frac{i-z}{i+z}} = z$$

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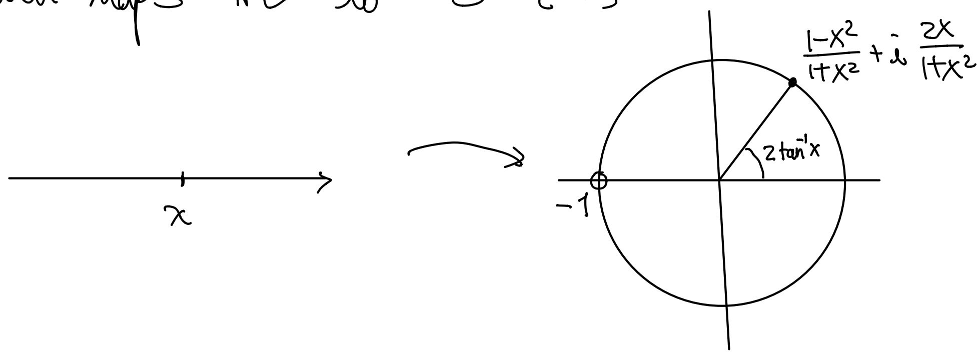
Remark: $F|_{\partial\mathbb{H}} : \mathbb{R} \to \partial D = S^1$ is continuous, and

$$F(x) = \frac{i-x}{i+x} = \frac{1-x^2}{1+x^2} + i\,\frac{2x}{1+x^2}$$

which maps $\mathbb{R}$ to $S^1 \setminus \{-1\}$



One should think of $F(\infty) = -1$. And $G(-1) = \infty$.

Def: Mappings of the form

$$z \mapsto \frac{az+b}{cz+d}, \qquad \left(\mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}\right)$$

with $a,b,c,d \in \mathbb{C}$ and $ad-bc \neq 0$,
are called <u>fractional linear transformations</u>

Remarks: (i) $ad-bc \neq 0 \iff cz+d \neq k(az+b)$ and $(az+b) \neq k(cz+d)$ (for some $k \in \mathbb{C}$)

$\iff z \mapsto \frac{az+b}{cz+d}$ is not a constant map.

(ii) Some other authors call them <u>linear fractional transformations</u>, or <u>Möbius transformations</u>.

1.2 Further examples

Eg 1 • Translations are conformal

$$z \mapsto z+h : \mathbb{C} \to \mathbb{C} \qquad (h \in \mathbb{C})$$

(Inverse $w \mapsto w-h$)

Remark: If $h \in \mathbb{R}$, then $z \mapsto z+h : \mathbb{H} \to \mathbb{H}$ is conformal.

• Dilations are conformal

$$z \mapsto cz : \mathbb{C} \to \mathbb{C} \qquad (c \in \mathbb{C} \setminus \{0\})$$

(Inverse $w \mapsto c^{-1}w$)

Remarks: (i) If $|c|=1$, then $c = e^{i\varphi}$ and

$$z \mapsto cz = e^{i\varphi} z : \mathbb{C} \to \mathbb{C} \text{ is a rotation}$$

When restricted $z \mapsto e^{i\varphi} z : \mathbb{D} \to \mathbb{D}$ is also conformal.

(ii) $c > 0$ $\quad z \mapsto cz$ is a (real) dilation

when restricted $z \mapsto cz : \mathbb{H} \to \mathbb{H}$ is conformal

(iii) $c < 0$ $\quad z \mapsto cz = -|c|z$ is a (real) dilation by $|c|$ followed by a rotation of angle π.

Note that translations and dilations are special cases of fractional linear transformations:

translations

$$z \mapsto z+h = \frac{z+h}{0 \cdot z + 1}$$

i.e. $a = 1 = d$, $b = h$, $c = 0$

& $ad - bc = 1 \neq 0$.

dilations $\quad z \mapsto cz \qquad c \neq 0$

$$= \frac{cz+0}{0\cdot z+1} \qquad \& \quad c\cdot 1 - 0\cdot 0 = c \neq 0.$$

Eg 1' (not in textbook)

(Complex) Inversion

$$z \mapsto \begin{cases} \frac{1}{z}, & \mathbb{C}\setminus\{0\} \to \mathbb{C}\setminus\{0\} \\ 0, & z = \infty \\ \infty, & z = 0 \end{cases}$$

is conformal.

Note that Inversion is also a fractional linear transformation

$$z \mapsto \frac{1}{z} = \frac{0\cdot z+1}{z+0} \qquad \& \quad 0\cdot 0 - 1\cdot 1 = -1 \neq 0.$$