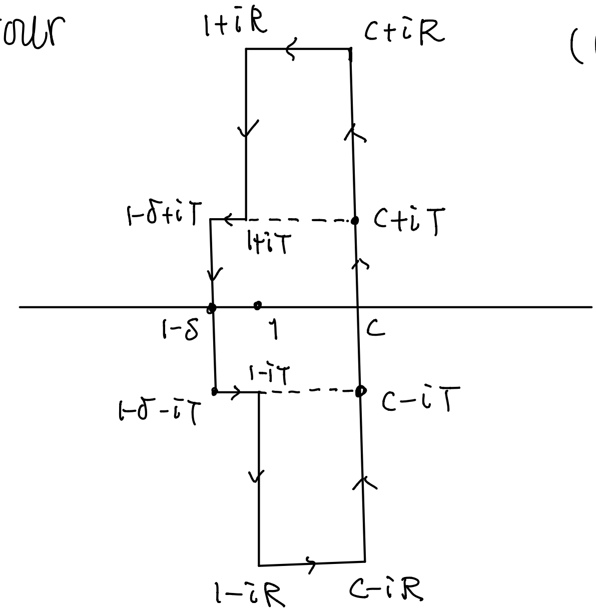
2.1 Proof of the asymptotics for ψ_1

(ie. Final step of the Proof of Prime Number Theorem)

Denote $F(s) = \dfrac{x^{s+1}}{s(s+1)}\left(-\dfrac{\zeta'(s)}{\zeta(s)}\right)$ where x fixed (& suff. large) say ≥ 2

Then Prop 2.3 $\Rightarrow$ $\forall c>0$, $\psi_1(x) = \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} F(s)\,ds$

Consider the contour $(R \to +\infty)$



where $T \geq 3$ and $\delta > 0$ is chosen (depending on T) such that $\zeta(s) \neq 0$ along the contour.

This can be done since $\zeta(s) \neq 0$ $\forall$ $\mathrm{Re}(s) \geq 1$ (Thm 1.1 & 1.2)

By Prop 2.7(ii) in Ch6 and Prop 1.6 in Ch7,

$\forall \varepsilon > 0$, $\exists$ $C_\varepsilon > 0$ s.t

$|\zeta'(s)| \leq C_\varepsilon |t|^\varepsilon$ and

$1/|\zeta(s)| \leq C_\varepsilon |t|^\varepsilon$ $\quad (\forall \sigma \geq 1$ & $|t| \geq 1)$

Hence $\forall \eta > 0$, $\exists A > 0$ s.t.

$$(*), \qquad \left| \frac{\zeta'(s)}{\zeta(s)} \right| \le A|t|^{\eta} \qquad \forall\, \sigma \ge 1 \ \& \ |t| \ge 1$$

$\Rightarrow$ For $R\ (>T)$ sufficiently large,

$$|F(s)| = \frac{|x^{s+1}|}{|s(s+1)|} \left| \frac{\zeta'(s)}{\zeta(s)} \right|$$

$$\le x^{c+1} \cdot \frac{1}{|s(s+1)|} \left| \frac{\zeta'(s)}{\zeta(s)} \right| \le A' |t|^{-2+\eta} \quad \text{for some } A' > 0$$

(A' indep. of s)

holds for all s on the horizontal line segments $[1+iR, c+iR]$ and $[1-iR, c-iR]$,

$$\Rightarrow \left| \int_{c+iR}^{1+iR} F(s)\, ds \right| \le A' R^{-2+\eta} (c-1) \to 0 \text{ as } R \to \infty$$

and $$\left| \int_{1-iR}^{c-iR} F(s)\, ds \right| \le A' R^{-2+\eta} (c-1) \to 0 \text{ as } R \to \infty$$

letting $R \to \infty$, Residue formula $\Rightarrow$

$$\operatorname{res}_{s=1} F(s) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)\, ds - \Bigg[\frac{1}{2\pi i} \left(\int_{1+iT}^{1+i\infty} + \int_{1-i\infty}^{1-iT} \right) F(s)\, ds + \frac{1}{2\pi i} \left(\int_{1-\delta+iT}^{1+iT} - \int_{1-\delta-iT}^{1-iT} \right) F(s)\, ds + \frac{1}{2\pi i} \int_{1-\delta-iT}^{1-\delta+iT} F(s)\, ds \Bigg]$$

By Cor 2.6 of Ch6,

$$\zeta(s) = \frac{1}{s-1} + H(s) \quad \text{near } s=1 \text{ with holo. } H(s).$$

$$\Rightarrow \quad -\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} - \underbrace{\frac{H(s)+(s-1)H'(s)}{1+(s-1)H(s)}}_{\text{holo. near } s=1}$$

$$\therefore \quad \operatorname{res}_{s=1} F(s) = \operatorname{res}_{s=1}\left[\frac{x^{s+1}}{s(s+1)}\cdot\left(\frac{1}{s-1} - h(s)\right)\right]$$

$$= \frac{x^2}{2}.$$

$$\therefore \quad \psi_1(x) = \frac{x^2}{2} + \frac{1}{2\pi i}\left(\int_{1+iT}^{1+i\infty} + \int_{1-i\infty}^{1-iT}\right)F(s)\,ds$$

$$+ \frac{1}{2\pi i}\left(\int_{1-\delta+iT}^{1+iT} - \int_{1-\delta-iT}^{1-iT}\right)F(s)\,ds + \frac{1}{2\pi i}\int_{1-\delta-iT}^{1-\delta+iT} F(s)\,ds$$

Since we care only the limit as $x \to +\infty$, we may assume $x \geq 2$ in our estimates.

(i) $$\left|\int_{1+iT}^{1+i\infty} F(s)\,ds\right| \leq \int_T^{\infty} \frac{|x^{2+it}|}{|(1+it)(2+it)|}\left|\frac{\zeta'(1+it)}{\zeta(1+it)}\right| dt$$

$$\leq x^2 \cdot \int_T^{\infty} \frac{1}{|1+it||2+it|}\cdot A|t|^{1/2}\,dt \quad \left(\text{take } \eta = \tfrac{1}{2} \text{ in } (*)_1\right)$$

Clearly the integral converges and hence

$\forall \varepsilon > 0$, $$\left|\frac{1}{2\pi i}\int_{1+iT}^{1+i\infty} F(s)\,ds\right| \leq \varepsilon\frac{x^2}{2} \text{ for suff. large } T.$$

Same argument $\Rightarrow$

$\forall \varepsilon > 0$, $\left| \frac{1}{2\pi i} \int_{1-i\infty}^{1-iT} F(s)\, ds \right| \le \varepsilon \frac{x^2}{2}$ for suff. large T.

(ii) $$\left| \frac{1}{2\pi i} \int_{1-\delta+iT}^{1+iT} F(s)\, ds \right| \le \frac{1}{2\pi} \int_{1-\delta}^{1} \frac{|x^{1+\sigma+iT}|}{|\sigma+iT|\,|\sigma+1+iT|} \left| \frac{\zeta'(\sigma+iT)}{\zeta(\sigma+iT)} \right| d\sigma$$

$$\le \frac{1}{2\pi} \int_{1-\delta}^{1} \frac{x^{1+\sigma}}{T^2} A T^{\frac{1}{2}}\, d\sigma \qquad (\eta = \tfrac{1}{2} \text{ in } (*),)$$

$$= C_T' \int_{1-\delta}^{1} x^{1+\sigma}\, d\sigma$$

$$= C_T' \int_{1-\delta}^{1} e^{(1+\sigma)\log x}\, d\sigma$$

$$= C_T' \left[\frac{e^{(1+\sigma)\log x}}{\log x} \right]_{1-\delta}^{1} \le C_T' \frac{x^2}{\log x}$$

$(x \ge 2 \Rightarrow \log x \ge \log 2 > 0)$

Similarly $\left| \frac{1}{2\pi i} \int_{1-\delta-iT}^{1-iT} F(s)\, ds \right| \le C_T' \frac{x^2}{\log x}$ for same const. $C_T' > 0$.

(iii) $$\left| \frac{1}{2\pi i} \int_{1-\delta-iT}^{1-\delta+iT} F(s)\, ds \right| \le \frac{1}{2\pi} \int_{-T}^{T} \frac{|x^{1+(1-\delta)+it}|}{|1-\delta+it|\,|2-\delta+it|} \left| \frac{\zeta'(1-\delta+it)}{\zeta(1-\delta+it)} \right| dt$$

$$\le \frac{1}{2\pi} \int_{-T}^{T} \frac{x^{2-\delta}}{|1-\delta+it|\,|2-\delta+it|} \left| \frac{\zeta'(1-\delta+it)}{\zeta(1-\delta+it)} \right| dt$$

$$\le C_T\, x^{2-\delta} \quad \text{for some const. } C_T$$

(depending on T, δ and hence depending on T as δ is chosen according to T)

Hence (i), (ii) & (iii) $\Rightarrow$ $\forall \varepsilon > 0$, $\exists \delta > 0$, C_T & C_T' s.t.

$$\left| \psi_1(x) - \frac{x^2}{2} \right| \le \varepsilon \frac{x^2}{2} + C_T' \frac{x^2}{\log x} + C_T x^{2-\delta}$$

for sufficiently large T

$$\Rightarrow \quad \left| \frac{2\psi_1(x)}{x^2} - 1 \right| \le \varepsilon + \underbrace{2C_T' \frac{1}{\log x} + 2C_T \frac{1}{x^\delta}}_{\longrightarrow 0 \text{ as } x \to \infty}.$$

Hence $\left| \frac{2\psi_1(x)}{x^2} - 1 \right| \le 4\varepsilon$ for sufficiently large x.

$$\Rightarrow \quad \lim_{x \to +\infty} \frac{2\psi_1(x)}{x^2} = 1$$

i.e. $\psi_1(x) \sim \frac{x^2}{2}$ as $x \to \infty$.

This completes the proof of prime number theorem. #