1.1 Analytic Continuation

> **Lemma 1.2** If $Re(s)>0$, then
> $$\Gamma(s+1) = s\Gamma(s). \quad \text{———} (2)$$
> Hence $\Gamma(n+1) = n!$ for $n=0,1,2,3,\cdots$

Pf: For $Re(s)>0$,

$$\Gamma(s+1) = \lim_{\varepsilon\to 0}\int_\varepsilon^{\frac{1}{\varepsilon}} e^{-t}t^s\,dt.$$

$$= \lim_{\varepsilon\to 0}\left\{\left[-e^{-t}t^s\right]_\varepsilon^{\frac{1}{\varepsilon}} + \int_\varepsilon^{\frac{1}{\varepsilon}} e^{-t}\cdot s t^{s-1}\,dt\right\}$$

$$= \lim_{\varepsilon\to 0}\left[\left(e^{-\varepsilon}\varepsilon^s - e^{-\frac{1}{\varepsilon}}\left(\frac{1}{\varepsilon}\right)^s\right) + s\int_\varepsilon^{\frac{1}{\varepsilon}} e^{-t}t^{s-1}\,dt\right]$$

$$= s\Gamma(s)$$

since $Re(s)>0 \Rightarrow$

$$\begin{cases} \left|e^{-\varepsilon}\varepsilon^s\right| = e^{-\varepsilon}\varepsilon^{Re(s)} \to 0 \\ \left|e^{-\frac{1}{\varepsilon}}\left(\frac{1}{\varepsilon}\right)^s\right| = e^{-\frac{1}{\varepsilon}}\left(\frac{1}{\varepsilon}\right)^{Re(s)} \to 0 \end{cases} \quad \text{as } \varepsilon\to\infty$$

This proves formula (2).

By formula (2), if $n\geq 1$, then

$$\Gamma(n+1) = n\Gamma(n) = \cdots = n(n-1)\cdots 1\cdot\Gamma(1)$$

$$= n!\,\Gamma(1).$$

And $\Gamma(1) = \int_0^\infty e^{-t} t^{1-1} dt = \int_0^\infty e^{-t} dt = 1$

$\therefore \Gamma(n+1) = n! \quad (n \geq 1)$

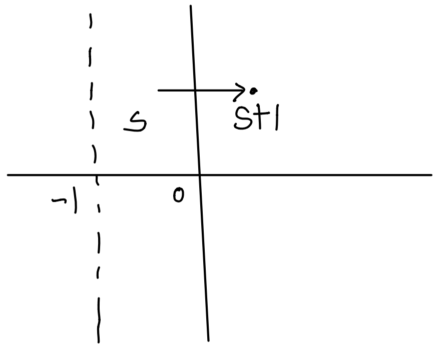
For $n=0$, $\quad \Gamma(0+1) = 1 = 0!$ by definition. #

<u>Thm 1.3</u> The gamma function $\Gamma(s)$ defined for $\mathrm{Re}(s) > 0$ has an <u>analytic continuation</u> to a <u>meromorphic</u> function on $\mathbb{C}$ whose only <u>singularities</u> are <u>simple poles</u> at $s = 0, -1, -2, \cdots$ with residue

$$\mathrm{res}_{s=-n} \Gamma(s) = \frac{(-1)^n}{n!}$$

<u>Remark</u>: Since $\mathbb{C} \setminus \{0, -1, -2, \cdots\}$ is connected, the analytic continuation of $\Gamma(s)$ is unique (by Thm 4.8 & Cor 4.9 of Ch 2). Therefore, it is convenient to denoted this analytic continuation again by $\Gamma(s)$.

So after proving this Theorem, the gamma function $\Gamma(s)$ is a <u>meromorphic function</u> on $\mathbb{C}$.



Pf: For $Re(s) > -1$, define

$$F_1(s) = \frac{\Gamma(s+1)}{s}$$

Since $\Gamma(s)$ holo in $Re(s) > 0$,

$\Gamma(s+1)$ holo in $Re(s) > -1$,

and hence $F_1(s) = \frac{\Gamma(s+1)}{s}$ is meromorphic in $Re(s) > -1$

with a simple pole at $s = 0$ with

$$\operatorname{res}_{s=0} F_1(s) = \Gamma(0+1) = 1.$$

Note that Lemma 1.2 $\Rightarrow$ $F_1(s) = \frac{\Gamma(s+1)}{s} = \Gamma(s)$ for $Re(s) > 0$,

$F_1(s)$ is an analytic continuation of $\Gamma(s)$ to a mero. function on $\{s \in \mathbb{C} : Re(s) > -1\}$.

Same argument works for $Re(s) > -m$ by defining

$$F_m(s) = \frac{\Gamma(s+m)}{(s+m-1)(s+m-2)\cdots s}.$$

Clearly $F_m(s)$ is meromorphic in $Re(s) > -m$ ($\Rightarrow Re(s+m) > 0$)

with simple poles at $s = 0, 1, \cdots, m-1$,

and for $n = 0, 1, \cdots, m-1$

$$\operatorname{res}_{s=-n} F_m(s) = \frac{\Gamma(-n+m)}{(-n+m-1)(-n+m-2)\cdots(1)(-1)(-2)\cdots(-n)}$$

the term corresponding to the pole

$$= \frac{\Gamma(m-n)}{(m-n-1)!\,(-1)^n n!}$$

$$= \frac{(-1)^n}{n!} \quad \text{by Lemma 1.2.}$$

And for $\operatorname{Re}(s) > 0$,

$$F_m(s) = \frac{\Gamma(s+m)}{(s+m-1)(s+m-2)\cdots s} = \frac{(s+m-1)\Gamma(s+m-1)}{(s+m-1)(s+m-2)\cdots s} \quad (\text{by Lemma 1.2})$$

$$= F_{m-1}(s) \cdots = F_1(s) = \Gamma(s)$$

$\therefore$ $F_m(s)$ is an analytic continuation of $\Gamma(s)$ to $\{\operatorname{Re}(s) > -m\}$.

Then uniqueness of theorem $\Rightarrow$ if $m > n$, then

$$F_m(s) = F_n(s) \quad \text{for } \operatorname{Re}(s) > -n.$$

Therefore, one can define meromorphic function $F(s)$ on $\mathbb{C}$ by

$$F(s) \overset{\text{def}}{=\!=} F_m(s) \quad \text{if } \operatorname{Re}(s) > -m.$$

Clearly, this gives the required analytic continuation. #

Remarks: (1) Clearly $\lim_{s\to 0} s\Gamma(s) = \Gamma(1) = 1$

(2) $\Gamma(s+1) = s\Gamma(s)$ holds for $s \in \mathbb{C}\setminus\{-1,-2,\cdots\}$.

Pf: LHS holo. except $s+1 = 0,-1,-2,\cdots$

RHS holo. except $s = -1,-2,\cdots$

since $s=0$ is a simple pole hence it is removable for RHS.

And on $\{Re(s)>0\}$, LHS = RHS. Therefore uniqueness thm $\Rightarrow$ LHS $\equiv$ RHS on $\mathbb{C}\setminus\{-1,-2,\cdots\}$.

(3) $\operatorname{res}_{s=-n} \Gamma(s+1) = -n \operatorname{res}_{s=-n} \Gamma(s) \quad (n=1,2,3,\cdots)$

Pf: near $s = -(n-1)$,

$$\Gamma(s) = \frac{\frac{(-1)^{n-1}}{(n-1)!}}{s+(n-1)} + \text{holo}(s)$$

$$\Rightarrow \quad \Gamma(s+1) = \frac{\frac{(-1)^{n-1}}{(n-1)!}}{s+n} + \text{holo}(s+1)$$

$$\therefore \quad \operatorname{res}_{s=-n} \Gamma(s+1) = \frac{(-1)^{n-1}}{(n-1)!} = -n \cdot \operatorname{res}_{s=-n} \Gamma(s).$$

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Alternating Proof of Thm 1.3: $\forall\, s \in \mathbb{C}\setminus\{0,-1,-2,\cdots\}$,

$$\Gamma(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\cdot\frac{1}{n+s} + \int_1^{\infty} e^{-t} t^{s-1}\,dt \qquad (3)$$

Pf: We 1st show (3) for $Re(s)>0$.

By Prop 1.1 and formula (1), for $\mathrm{Re}(s)>0$,

$$\Gamma(s)=\int_0^\infty e^{-t}t^{s-1}\,dt$$
$$=\int_0^1 e^{-t}t^{s-1}\,dt+\int_1^\infty e^{-t}t^{s-1}\,dt$$

For $t\in(0,1)$, $e^{-t}=\sum_{n=0}^\infty \frac{(-t)^n}{n!}=\sum_{n=0}^\infty \frac{(-1)^n}{n!}t^n$

By absolute convergence of the improper integral and uniform convergence of the series, we have

$$\int_0^1 e^{-t}t^{s-1}\,dt=\int_0^1\left(\sum_{n=0}^\infty \frac{(-1)^n}{n!}t^n\right)t^{s-1}\,dt$$
$$=\sum_{n=0}^\infty \frac{(-1)^n}{n!}\int_0^1 t^{n+s-1}\,dt$$
$$=\sum_{n=0}^\infty \frac{(-1)^n}{n!}\cdot\frac{1}{n+s}$$

$\therefore$ $$\Gamma(s)=\sum_{n=0}^\infty \frac{(-1)^n}{n!}\cdot\frac{1}{n+s}+\int_1^\infty e^{-t}t^{s-1}\,dt,\quad \forall\ \mathrm{Re}\,s>0.$$

Now clearly $\int_1^\infty e^{-t}t^{s-1}dt$ an <u>entire</u> function because of the exponential decay (Ex!).

For the series, consider any $R>0$ and any $N>2R$

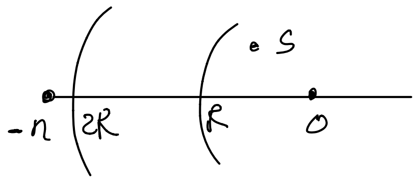
$$\sum_{n=0}^\infty \frac{(-1)^n}{n!}\cdot\frac{1}{n+s}=\sum_{n=0}^N \frac{(-1)^n}{n!}\cdot\frac{1}{n+s}+\sum_{n=N+1}^\infty \frac{(-1)^n}{n!}\cdot\frac{1}{n+s}$$

$\sum_{n=0}^{N} \frac{(-1)^n}{n!} \cdot \frac{1}{n+s}$ is a meromophic function in $\{|s|<R\}$

with poles at $k \in \{0,-1,-2,\cdots,-N\}$ such that $|k|<R$.

$\sum_{n=N+1}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{1}{n+s}$ has general term

$$\left| \frac{(-1)^n}{n!} \cdot \frac{1}{n+s} \right| \leq \frac{1}{n!} \cdot \frac{1}{R}$$



since $n > N > 2R$ and $|s|<R$

$\Rightarrow \quad |n+s| > R$

$\therefore \quad \sum_{n=N+1}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{1}{n+s}$ uniformly converges to a holomorphic

function in $\{|s|<R\}$.

Since $R>0$ is arbitrary,

$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{1}{n+s} + \int_1^{\infty} e^{-t} t^{s-1} dt$ defines a meromophic

function with simple poles at $s=\{0,-1,-2,\dots\}$

with $\operatorname{res}_{s=-n} = \frac{(-1)^n}{n!}$.

Since it $= \Gamma(s)$ for $\operatorname{Re} s>0$, we've proved (3), $\forall\, s \in \mathbb{C} \setminus \{0,-1,\dots\}$.

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