§4 Weierstrass Infinite Products

Thm 4.1 Given any seq $\{a_n\} \subset \mathbb{C}$ with $|a_n| \to +\infty$ as $n \to +\infty$, $\exists$ entire function f such that

$$\begin{cases} f(a_n) = 0, & \forall n, \\ f(z) \neq 0, & \forall z \in \mathbb{C} \setminus \{a_n\} \end{cases}$$

If g is another entire function with the same property, then $\exists$ entire function $h(z)$ such that

$$g(z) = f(z)\, e^{h(z)}.$$

Pf: The 2^{nd} statement is easy to prove: near $z = a_n$

$$\frac{g(z)}{f(z)} = \frac{(z-a_n)^m g_1(z)}{(z-a_n)^m f_1(z)}$$ where f_1, g_1 holo. near a_n and $f_1(a_n) \neq 0, g_1(a_n) \neq 0$

$$= \frac{g_1(z)}{f_1(z)}$$ holo. near a_n

$\therefore \{a_n\}$ are removable singularities of $\dfrac{g(z)}{f(z)}$

Since f & g have no other zeros, $g(z)/f(z)$ is entire with no zero. Therefore $\dfrac{g(z)}{f(z)} = e^{h(z)}$ for some entire function $h(z)$. ※

To prove the 1^{st} statement, we need a lemma concerning <u>canonical factors</u>:

$$\begin{cases} E_0(z) = 1 - z \quad \& \\ E_k(z) = (1-z)\, e^{z + \frac{z^2}{2} + \cdots + \frac{z^k}{k}}, \quad k \geq 1 \end{cases}$$

(k = degree of the canonical factor)

Lemma 4.2 $\exists\, C>0$ such that $\forall k\geq 0$,

$$|1-E_k(z)| \leq C|z|^{k+1} \quad \text{for } z\in \overline{D_{\frac{1}{2}}(0)}$$

Pf: On $\overline{D_{\frac{1}{2}}(0)}$, $\log(1-z)$ well-defined

$$\Rightarrow \quad E_k(z) = (1-z)e^{z+\frac{z^2}{2}+\cdots+\frac{z^k}{k}}$$

$$= e^{\log(1-z)+z+\frac{z^2}{2}+\cdots+\frac{z^k}{k}}$$

$j=n-k-1$

By Taylor's expansion of $\log(1-z)$,

$$\log(1-z)+z+\frac{z^2}{2}+\cdots+\frac{z^k}{k} = -\sum_{n=k+1}^{\infty}\frac{z^n}{n} = -z^{k+1}\sum_{n=k+1}^{\infty}\frac{z^{n-k-1}}{n}$$

$$= -z^{k+1}\sum_{j=0}^{\infty}\frac{z^j}{k+1+j}$$

Denote the LHS by w, then

$$|w| \leq |z|^{k+1}\sum_{j=0}^{\infty}|z|^j \leq |z|^{k+1}\sum_{j=0}^{\infty}\left(\frac{1}{2}\right)^j = 2|z|^{k+1} \left(\leq \frac{1}{2^k}\right)$$

$$\therefore \quad |1-E_k(z)| = |1-e^w| \leq c'|w| \quad \text{for some } c'>0 \text{ (indep. of } k)$$

$$\leq 2c'|z|^{k+1}. \quad \#$$

Pf of the 1st statement of Thm 4.1

If 0 is a "m-order zero" of f (m could be 0, ie $f(0)\neq 0$)

we remove those $a_{n_1}=\cdots a_{n_m}=0$ from the seq $\{a_n\}$.

For simplicity, denote the subseq. by $\{a_n\}$ again.

Then consider the infinite product.

$$f(z) = z^m \prod_{n=1}^{\infty} E_n\left(\frac{z}{a_n}\right).$$

For any fixed $R > 0$, by re-arranging finitely many terms, we may assume $|a_n| \leq 2R$ for $n = 1, \cdots, n_0 - 1$

and $|a_n| > 2R$ for $n \geq n_0$

(as $|a_n| \to +\infty$)

$\forall z \in D_R$, we have $\left|\frac{z}{a_n}\right| < \frac{1}{2}$ for $n \geq n_0$

By Lemma 4.2, $\left|1 - E_n\left(\frac{z}{a_n}\right)\right| \leq C\left|\frac{z}{a_n}\right|^{n+1}$ for some $C > 0$ indep. of n

$$\leq \frac{C}{2^{n+1}}$$

$\Rightarrow \sum_{n=n_0}^{\infty} \left|1 - E_n\left(\frac{z}{a_n}\right)\right|$ is convergent.

Hence Prop 3.2 $\Rightarrow \prod_{n=n_0}^{\infty} E_n\left(\frac{z}{a_n}\right) = \prod_{n=n_0}^{\infty} \left[1 + \left(E_n\left(\frac{z}{a_n}\right) - 1\right)\right]$

converges uniformly on D_R

$\Rightarrow \prod_{n=n_0}^{\infty} E_n\left(\frac{z}{a_n}\right)$ is a holo. function on D_R

and Prop 3.1 $\Rightarrow \prod_{n=n_0}^{\infty} E_n\left(\frac{z}{a_n}\right) \neq 0 \quad \forall z \in D_R$

$$\therefore \quad f(z) = z^m \prod_{n=1}^{\infty} E_n\left(\frac{z}{a_n}\right) = z^m \prod_{n=1}^{n_0-1} E_n\left(\frac{z}{a_n}\right) \cdot \prod_{n=n_0}^{\infty} E_n\left(\frac{z}{a_n}\right)$$

if $m > 0$

is holo on D_R with only those zeros at $z = 0$ or $z = a_n$ with $|a_n| < R$.

As $R>0$ is arbitrary, $f(z)$ converges locally uniformly to an entire function with exactly the zeros prescribed by the sequence $\{a_n\}$ (and m-order zero at 0) #