Ch2 Cauchy's Theorem & Its applications

§1 Goursat's Theorem

Thm 1.1 & Cor 1.2

If
- Ω open in $\mathbb{C}$,
- f holomorphic on Ω,
- γ = boundary of a triangle T or rectangle R s.t. $\gamma \cup T$ or $\gamma \cup R \subset \Omega$,

(note the different in terminology in the text book)

then $\int_\gamma f(z)\,dz = 0$

Remark: The main point in Goursat's Thm is that there is no need to assume f' is continuous. Cauchy's first observation used Green's Thm which need to assume u_x, u_y, v_x & v_y are continuous.

§2 Local existence of primitive & Cauchy's Theorem in a disc

(and Appendix B: Simply Connectivity and Jordan Curve Theorem)

Notation: For a simple closed piecewise smooth curve γ,

$\mathrm{int}(\gamma)$ = bounded component of $\mathbb{C} \setminus \gamma$

(i.e. the interior of the Jordan curve of γ, not the interior of γ as a topological point set)

Thm 2.9 (on page 361 of the text book)

If
- $f : \Omega \to \mathbb{C}$ is holo., Ω open,
- γ = simple closed piecewise smooth curve s.t.
- $\gamma \cup \operatorname{int}(\gamma) \subset \Omega$

Then
$$\int_\gamma f \, dz = 0.$$

§3 Evaluation of some integrals (self reading)

§4 Cauchy's Integral Formula

Thm 4.1 & Cor 4.2

If
- f is holo. on Ω.
- C positive oriented simple closed piecewise smooth curve s.t.
- $C \cup \operatorname{int}(C) \subset \Omega$

then $\forall\ z \in \operatorname{int}(C)$ & $n = 0, 1, 2, \cdots$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta.$$

<u>Consequences</u>

- <u>Cor 4.3</u> Cauchy inequalities
- <u>Thm 4.4</u> Holomorphic $\Rightarrow$ analytic & Taylor's formula
- <u>Cor 4.5</u> Liouville's Theorem
- <u>Cor 4.6</u> Foundamental Theorem of Algebra
- <u>Cor 4.7</u> Factorization of Polynomial
- <u>Thm 4.8</u> & <u>Cor 4.9</u> Uniqueness of holomorphic function

Self reading.

§5 <u>Further applications</u>

5.1 <u>Morera's Thm</u> (Converse of Cauchy's Thm)

> <u>Thm 5.1</u>
> - f cts. on Ω &
> - $\int_{\partial T} f = 0$ $\forall$ triangle T with $T \cup \partial T \subset \Omega$,
>
> (note the diff. in terminology in the textbook)
>
> then f is holomorphic on Ω.

5.2 <u>Sequence of Holomorphic Functions</u>

> <u>Thm 5.2 & Thm 5.3</u> f_n holo. on Ω, $f_n \to f$ uniformly on cpt. subsets
>
> Then f holo on Ω and $f_n' \to f'$ uniformly on cpt. subsets.

5.3 Holomorphic functions defined in terms of integrals

Thm 5.4 Ω open in $\mathbb{C}$, $F(z,s): \Omega \times [a,b] \to \mathbb{C}$.

Suppose (1) For each $s \in [a,b]$, $F(z,s)$ is holo. in z.

(2) $F \in C(\Omega \times [a,b])$

Then

$$f(z) = \int_a^b F(z,s)\,ds$$

is a holomorphic function on Ω.

(The proof is not covered in MATH2230)

Pf: It is clear that one may assume $[a,b] = [0,1]$.

Since Ω may be unbounded, we work on an arbitrary disc $D \subset \overline{D} \subset \Omega$.

For $n \geq 1$, consider Riemann sum

$$f_n(z) = \frac{1}{n} \sum_{k=1}^{n} F(z, \tfrac{k}{n})$$

Then, (i) $\Rightarrow$ $f_n(z)$ is holo. $\forall n \geq 1$.

By (ii), $F \in C(\Omega \times [0,1])$

$\Rightarrow$ $F(z,s)$ is uniformly continuous on $\overline{D} \times [0,1]$,

$\Rightarrow$ $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall z \in \overline{D}$

$$|F(z,s_1) - F(z,s_2)| < \varepsilon, \quad \forall |s_1 - s_2| < \delta$$

(since $\operatorname{dist}((z,s_1),(z,s_2)) = |s_1 - s_2| < \delta$)

$$\Rightarrow \sup_{z\in D} |F(z,s_1)-F(z,s_2)|<\varepsilon, \quad \forall\, |s_1-s_2|<\delta.$$

Therefore, if $n>\frac{1}{\delta}$, then $\forall z\in D\subset\overline{D}$

$$|f_n(z)-f(z)| = \left|\frac{1}{n}\sum_{k=1}^{n} F(z,\tfrac{k}{n}) - \int_0^1 F(z,s)\,ds\right|$$

$$= \left|\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}} F(z,\tfrac{k}{n})\,ds - \int_0^1 F(z,s)\,ds\right|$$

$$= \left|\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}} \left(F(z,\tfrac{k}{n})-F(z,s)\right)ds\right|$$

$$\leq \sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}} \left|F(z,\tfrac{k}{n})-F(z,s)\right|ds$$

$$< \varepsilon\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}} ds = \varepsilon$$

$\therefore$ f is the uniform limit of f_n on D

By Thm 5.2 & 5.3, f is holomorphic on D.
Since $D\subset\overline{D}\subset\Omega$ is arbitrary, f is holomorphic on Ω.

$\#$

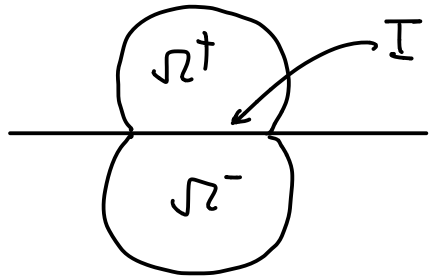
5.4 Schwarz reflection principle

Def: • An open set Ω in $\mathbb{C}$ is symmetric with respect to the real line if

$$z\in\Omega \Leftrightarrow \bar{z}\in\Omega.$$

If Ω is symmetric wrt $\mathbb{R}$-line, we denote

$$\Omega^+ = \{z = x+iy \in \Omega : y > 0\}$$
$$\Omega^- = \{z = x+iy \in \Omega : y < 0\}.$$



$$I = \Omega \cap \mathbb{R}.$$

Then $\Omega = \Omega^+ \cup I \cup \Omega^-$

Thm 5.5 (Symmetry principle)

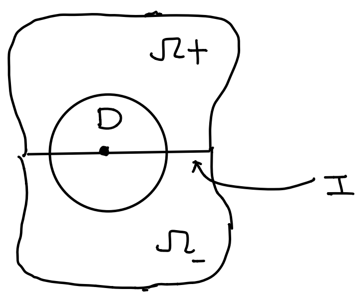
If $f^+ : \Omega^+ \to \mathbb{C}$, $f^- : \Omega^- \to \mathbb{C}$ holo. such that $f^\pm$ extend continuously to $\Omega^\pm \cup I$ with

$$f^+(x) = f^-(x), \ \forall x \in I,$$

then $f(z) = \begin{cases} f^+(z), & z \in \Omega^+ \\ f^+(z) = f^-(z), & z \in I \\ f^-(z), & z \in \Omega^- \end{cases}$ is holo. on Ω.

Pf: Clearly only need to show that f is holo at points of I.

Hence, we only need to consider a disc $D \subset \bar{D} \subset \Omega$ s.t. its center $\in I$.

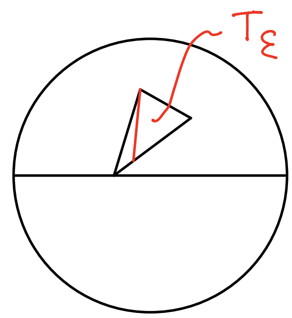


Then D is symmetric wrt $\mathbb{R}$-line too

Consider triangle $T \subset D$, if $T \subset D^+$ or D^-, then Cauchy's Thm $\Rightarrow \int_{\partial T} f\,dz = 0$.

If $T \cap I \neq \phi$, then

<u>Case 1</u> $T \cap I =$ a vertex of T

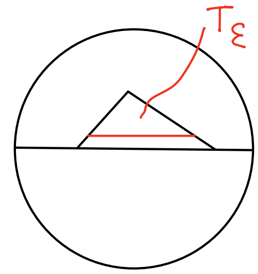


Approximate by a $T_\varepsilon \subset D^+ \text{ or } D^-$

Then uniform continuity of f &

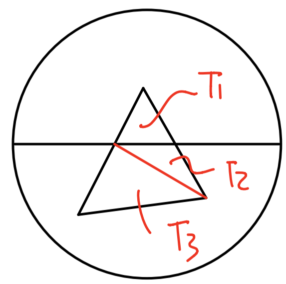
$\int_{\partial T_\varepsilon} f dz = 0, \forall \varepsilon > 0 \implies \int_{\partial T} f dz = 0$

<u>Case 2</u> $T \cap I =$ an edge of T



Same argument as in Case 1.

<u>Case 3</u> $T \cap D^+ \neq \phi$ and $T \cap D^- \neq \phi$



Then $T \cap I$ divides T into triangle or polygon completely contained in $D^+ \cup I$ or $D^- \cup I$. If it is a triangle, apply Case 2.
If it is a polygon, subdivide the polygon into triangles as in Cases 1 & 2. Then using results in cases 1 & 2 and by the cancellation of the integrals along the common edges, we have

$$\int_{\partial T} f dz = 0.$$

By Morera's Thm (Thm 5.1), f is holo. on I. #

Thm 5.6 (<u>Schwarz Reflection Principle</u>)

Let • Ω be symmetric wrt $\mathbb{R}$-line.

• $f: \Omega^+ \to \mathbb{C}$ is holomorphic and extends continuously to I such that

• $f(x) \in \mathbb{R}$, $\forall x \in I$.

Then $\exists F: \Omega \to \mathbb{C}$ holomorphic such that
$$F|_{\Omega^+} = f.$$

(In fact, F is unique by Thm 4.8 (assuming connectedness of Ω))

Pf: Define $f^-(z) = \overline{f(\bar{z})}$ for $z \in \Omega^-$.

Then it is easy to check

- $f^-: \Omega^- \to \mathbb{C}$ is holomorphic
- f^- extends continuously to I

and $\forall x \in I$, $f^-(x) = \overline{f(\bar{x})} = \overline{f(x)} = f(x)$ as $f(x) \in \mathbb{R}$

By Thm 5.5 (Symmetric principle)

$$F(z) = \begin{cases} f(z), & z \in \Omega^+ \cup I \\ f^-(z) = \overline{f(\bar{z})}, & z \in \Omega^- \end{cases}$$ is holomorphic on Ω.

and clearly $F|_{\Omega^+} = f$. ※

§5.5 <u>Runge's Approximation Theorem</u>

Omitted.

Ch3 Meromorphic Functions and the Logarithm

§1 Zeros and Poles

Thm 1.1 & Thm 1.2 Ω open in $\mathbb{C}$, $z_0 \in \Omega$, f holomorphic in $\Omega \setminus \{z_0\}$.
In a nbd. of z_0, $\exists$ holo function g and integer $n \geq 1$ s.t.

$$f(z) = \begin{cases} (z-z_0)^n g(z) & \Leftrightarrow \quad z_0 \text{ is a zero} \\ (z-z_0)^{-n} g(z) & \Leftrightarrow \quad z_0 \text{ is a pole} \end{cases}$$

(In case of zero, f is actually holo in Ω)

- multiplicity of zeros and poles
- simple zero and simple poles
- Laurent series expansion $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$, isolated singularities
- Principal part at a pole
- Residue at a pole

$$f(z) = \underbrace{\frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{-n+1}} + \cdots + \overset{\text{residue}}{\frac{a_{-1}}{z-z_0}}}_{\text{principal part}} + \underset{\text{holo in a nbd of } z_0}{G(z)}$$

§2 The Residue Formula

Thm 2.1, Cor 2.2 & Cor 2.3 (Residue formula)
Suppose f holo in an open set containing a simple closed piecewise smooth (+ve oriented) curve γ and $\text{int}(\gamma)$, except for poles at $z_1, \cdots, z_N \in \text{int}(\gamma)$. Then

$$\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{N} \text{res}_{z_k} f$$

§3 Singularities and meromorphic functions

Thm 3.1 (Riemann's Theorem on removable singularities)

Suppose Ω open in $\mathbb{C}$, $z_0 \in \Omega$.

$f: \Omega \setminus \{z_0\} \to \mathbb{C}$ holomorphic.

If f is bounded on $\Omega \setminus \{z_0\}$, then z_0 is a removable singularity (i.e. f can be extended to a holomorphic function on Ω)

$\Rightarrow$ For isolated singularities either
- removable (f bdd near z_0)
- pole ($|f| \to +\infty$ as $z \to z_0$)

or
- essential singularities.

Thm 3.3 (Casorati-Weierstrass)

If $f: D_r(z_0) \setminus \{z_0\} \to \mathbb{C}$ holo. and has an essential singularity at z_0, then

$f(D_r(z_0) \setminus \{z_0\})$ dense in $\mathbb{C}$.

- extended complex plane,
- rational functions
- Riemann sphere
- Stereographic projection

$\Bigg\}$ self-reading.

§4 The argument principle and applications

Thm 4.1 & Cor 4.2 (Argument Principle) (+ve oriented)

Suppose f mero. in an open set containing a simple closed piecewise smooth curve γ and $\mathrm{int}(\gamma)$. If f has neither zeros nor poles on γ, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = Z - P$$

where Z = number of zeros of f in $\mathrm{int}(\gamma)$ &
P = number of poles of f in $\mathrm{int}(\gamma)$

Thm 4.3 (Rouché's Theorem)

Suppose f & g are holo in an open set containing a simple closed piecewise smooth curve γ and $\mathrm{int}(\gamma)$. If

$$|f(z)| > |g(z)| \quad \forall z \in \gamma,$$

then f and $f+g$ have the same number of zeros in $\mathrm{int}(\gamma)$.

Thm 4.4 (Open Mapping Theorem)

If f holo on a region Ω & $f \not\equiv$ const., then f is open.
(i.e. f maps open sets to open sets.)

Thm 4.5 (Maximum modulus principle)

If f holo on a region Ω & $f \not\equiv$ const., then $|f|$ cannot attain a maximum in Ω.

For simplicity, sometimes we just say "maximum of f" for "maximum of $|f|$".
holo

Cor 4.6 Suppose Ω is a region with compact closure $\overline{\Omega}$. If f holo. on Ω & continuous on $\overline{\Omega}$, then

$$\sup_{z\in\Omega}|f(z)| \leq \sup_{z\in\overline{\Omega}\setminus\Omega}|f(z)|$$

§5 Homotopies and Simply Connected Domains

Def: Ω open in $\mathbb{C}$; $\gamma_0(t)$ & $\gamma_1(t)$, $t\in[a,b]$, curves in Ω with common end points, i.e. $\gamma_0(a)=\gamma_1(a)=\alpha$; $\gamma_0(b)=\gamma_1(b)=\beta$.
γ_0 & γ_1 are said to be homotopic in Ω if $\exists$ continuous map $H(s,t): [0,1]\times[a,b]\to\Omega$ such that

$$H(0,t)=\gamma_0(t) \quad \& \quad H(1,t)=\gamma_1(t), \quad \forall t\in[a,b].$$

and $$H(s,a)=\alpha \quad \& \quad H(s,b)=\beta, \quad \forall s\in[0,1]$$

Remark: Usually think of $H(s,t)=\gamma_s(t)$ as a family of curves in Ω with the same end points $\gamma_s(a)=\alpha$ & $\gamma_s(b)=\beta$
s.t. for $s=0$ & 1 are the original two curves γ_0 & γ_1.
Hence "one curve can be deformed continuously into the other curve without ever leaving Ω".

Thm 5.1 If f holo. in Ω, then

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz$$

provided γ_0 and γ_1 are homotopic in Ω.