Review (Ch1–3 of the Textbook)

Ch1 Preliminaries to Cpx Analysis

§1 Cpx numbers & Cpx plane (Self reading)

Recall notations:

- open disc of radius r centered at z_0 : $D_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$
- closed disc of radius r centered at z_0 : $\overline{D}_r(z_0) = \{z \in \mathbb{C} : |z - z_0| \leq r\}$
- boundary of $D_r(z_0)$ (or $\overline{D}_r(z_0)$) : $C_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}$

- unit disc : $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$

- diameter of a set $\Omega \subset \mathbb{C}$: $\operatorname{diam}(\Omega) = \sup_{z, w \in \Omega} |z - w|$

- region = open connected set in $\mathbb{C}$

§2 Functions of the Cpx plane

2.1 Self reading

2.2 Holomorphic functions

- Ω open set in $\mathbb{C}$,
- f cpx-valued function on Ω.

Def: f is holomorphic at the point $z_0 \in \Omega$ if

$$\lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h} \text{ exists.}$$

$(h \in \mathbb{C},\ h \neq 0 \text{ s.t. } z_0 + h \in \Omega)$

And if it exists, it is called the derivative of f at z_0

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

- f is said to be holomorphic on Ω if f is holomorphic at z_0, $\forall z_0 \in \Omega$.

- If C is a closed set in $\mathbb{C}$, then f is holomorphic on C if $\exists$ open set Ω s.t. $C \subset \Omega$ and f is holomorphic on Ω.

- f is called entire if f is holomorphic on $\mathbb{C}$

- Cauchy-Riemann equations

If $f = u + iv$ holomorphic on Ω (open), (u, v $\mathbb{R}$-valued) then

$$\boxed{\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \quad \text{on } \Omega}$$

- cpx differential operators $\frac{\partial}{\partial z}$ & $\frac{\partial}{\partial \bar{z}}$:

$$\boxed{\begin{aligned} \frac{\partial}{\partial z} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \\ \frac{\partial}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \end{aligned}}$$

- Then $\boxed{\text{Cauchy-Riemann} \Leftrightarrow \frac{\partial f}{\partial \bar{z}} = 0}$.

<u>Prop 2.3</u> $f = u + iv$ holomorphic at z_0, then

$$\begin{cases} \frac{\partial f}{\partial \bar{z}}(z_0) = 0 \\ \frac{\partial f}{\partial z}(z_0) = f'(z_0) = 2\frac{\partial u}{\partial z}(z_0) \end{cases}$$

Also $F : \Omega \to \mathbb{R}^2 : (x,y) \mapsto (u(x,y), v(x,y))$ is <u>differentiable</u>

and $\det J_F(x_0,y_0) = |f'(z_0)|^2$, (as $\Omega \to \mathbb{R}^2$ mapping)

where J_F is the <u>Jacobian matrix</u> of F

<u>Thm 2.4</u> $f = u + iv$ defined on an <u>open</u> $\Omega \subset \mathbb{C}$,

(u, v are <u>real-valued</u> functions on Ω)

If <u>$u, v \in C^1(\Omega)$</u> and satisfy <u>Cauchy-Riemann</u> eqt.

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \quad \text{on } \Omega.$$

then f is <u>holomorphic on Ω</u> & $f' = \frac{\partial f}{\partial z}$.

2.3 Power series $\sum_{n=0}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$

- absolute convergence (at z) if the real-valued series

$$\sum_{n=0}^{\infty} |a_n||z|^n \text{ converges}$$

Thm 2.5 Given $\sum_{n=0}^{\infty} a_n z^n$, define

$$R = \frac{1}{\limsup |a_n|^{\frac{1}{n}}} \quad (\in [0,\infty])$$

then (i) If $|z|<R$, $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely

(ii) If $|z|>R$, $\sum_{n=0}^{\infty} a_n z^n$ diverges

Remarks:
- no conclusion on $\{|z|=R\}$
- R is called the radius of convergence
- $\{|z|<R\}$ the disc of convergence

Thm 2.6

$f(z) = \sum_{n=0}^{\infty} a_n z^n$ holomorphic on the disc of convergence (provided $R>0$)

and

$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$ with the same radius of convergence.

Cor 2.7. $\sum_{n=0}^{\infty} a_n z^n$ infinitely (cpx) differentiable & higher derivatives can be calculated by termwise differentiation (in its disc of convergence)

Def $f: \Omega^{\text{(open)}} \to \mathbb{C}$ is analytic at $z_0 \in \Omega$
if $\exists \sum_{n=0}^{\infty} a_n (z-z_0)^n$ with positive radius of convergence
such that
$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{in a nbd. of } z_0.$$

Clearly by Thm 2.6, $\boxed{\text{holomorphic} \Leftarrow \text{analytic}}$

§3 Integration along curves : Self reading.

$$\int_\gamma f(z)dz$$

Useful notation : $\begin{cases} dz = dx + i\,dy \\ d\bar{z} = dx - i\,dy \end{cases}$

Then • $\int_\gamma f\,dz = \int_\gamma (u+iv)(dx+idy)$
$$= \int_\gamma (u\,dx - v\,dy) + i\int_\gamma (v\,dx + u\,dy)$$

• $df = du + i\,dv$
$$= f_x\,dx + f_y\,dy$$
$$= \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}$$

$(\therefore f \text{ holo.} \Rightarrow df = f' dz)$