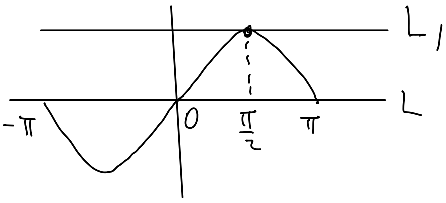
eg $f(x) = \sin x$

$L(x) \equiv 0$ (zero function)

Clearly L crosses f

(at $0, \pm\pi, \pm 2\pi, \cdots$)

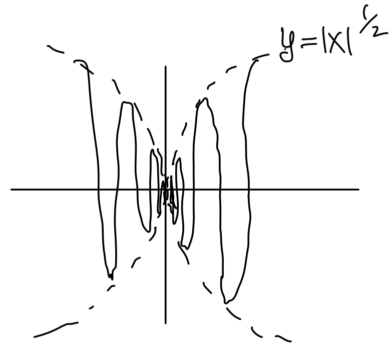
(At $x_0 = 0$, the $\delta > 0$ can be chosen as π)



If $L_1(x) \equiv 1$, L_1 doesn't cross f:

at every intersection $(2n+1)\frac{\pi}{2}$, $f(x) < L_1(x) \equiv 1$ for all $x \in \left((2n+1)\frac{\pi}{2} - \delta, (2n+1)\frac{\pi}{2} + \delta\right) \setminus \{(2n+1)\frac{\pi}{2}\}$, $\forall\, 0 < \delta < 2\pi$

eg: $f(x) = \begin{cases} |x|^{\frac{1}{2}} \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$



Then no "line" L crosses f at $x_0 = 0$

All "line" L passing thro $(0,0)$ intersects $y = \pm|x|^{\frac{1}{2}}$.

Then infinite oscillation of $f \Rightarrow$ neither (i) nor (ii) in the definition holds.

Def: A function $f: [a,b] \to \mathbb{R}$ is said to be "crosses no lines" if there is no $L(x) = \alpha x + \beta$ crosses f.

Thm The set $Z_1 = \{f \in C[a,b] : f \text{ crosses no lines}\}$ is a residual set in $C[a,b]$, and hence dense.

Pf: Note that

$$C[a,b]\setminus Z = \{ f \in C[a,b] : \exists \text{ some } L \text{ crosses } f \text{ (at some pt.)} \}$$

where $L(x) = \alpha x + \beta \quad (\alpha, \beta \in \mathbb{R})$

And we need to show that $C[a,b]\setminus Z$ is of 1^{st} category.

Notation: For $f \in C[a,b]$ and $\alpha \in \mathbb{R}$, we denote

$$f_{-\alpha}(x) = f(x) - \alpha x$$

(subtracting the linear part of L from f)

Let A_n be the set of $f \in C[a,b]$ for which $\exists\, \alpha \in [-n, n]$ and $x \in [a,b]$ such that

$$\begin{cases} f_{-\alpha}(t) \leq f_{-\alpha}(x) & \forall\, t \in (x - \frac{1}{n}, x) \\ f_{-\alpha}(t) \geq f_{-\alpha}(x) & \forall\, t \in (x, x + \frac{1}{n}) \end{cases} \qquad (t \in [a,b])$$

Clearly $A_n \subset A_{n+1}$, $\forall n$ $\quad$ (since $(x - \frac{1}{n+1}, x + \frac{1}{n+1}) \subset (x - \frac{1}{n}, x + \frac{1}{n})$)

Note that t is now the independent variable, and $f_{-\alpha}(t) \leq f_{-\alpha}(x)$ is exactly

$$f(t) \leq \alpha t + (f(x) - \alpha x) = L t \quad , \quad \forall\, t \in (x - \tfrac{1}{n}, x)$$

Similarly, $f_{-\alpha}(t) \geq f_{-\alpha}(x)$ is exactly

$$f(t) \geq \alpha t + (f(x) - \alpha x) = L t \quad , \quad \forall\, t \in (x, x + \tfrac{1}{n})$$

$\therefore$ $f \in A_n \Rightarrow f$ crosses L at x

(with $\delta < \frac{1}{n}$ and slope $|\alpha| \leq n$)

And if f crosses some L at some x, then

$f \in A_n$ or $-f \in A_n$ for some n

$\Rightarrow f \in A = \bigcup_{n=1}^{\infty} A_n$ or $-f \in A = \bigcup_{n=1}^{\infty} A_n$ (to be con't)