Pf: (a) let $E \subset X$ be a set of 1st category.

Then $E = \bigcup_{n=1}^{\infty} E_n$ for some nowhere dense sets E_n, $n=1,2,\dots$

let $F \subset E$, then by Prop 4.7 (a)

$F \cap E_n$ is nowhere dense, $\forall n$ $(F \cap E_n \subset E_n)$

Hence $F = F \cap E = \bigcup_{n=1}^{\infty} (F \cap E_n)$ is of 1st category.

(b) Let $E_n = \bigcup_{k=1}^{\infty} E_{n,k}$, $E_{n,k}$ = nowhere dense.

$\Rightarrow \bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} \left(\bigcup_{k=1}^{\infty} E_{n,k} \right) = \bigcup_{(n,k) \in \mathbb{N} \times \mathbb{N}} E_{n,k}$

is of 1st category. (since $\mathbb{N} \times \mathbb{N}$ is countable)

(c) If $E = \{x_i\}_{i=1}^{\infty} \subset X$, then Prop 4.7 (c)

$\Rightarrow$ $\{x_i\}$ is nowhere dense $\forall i$

$\Rightarrow$ $E = \bigcup_{i=1}^{\infty} \{x_i\}$ is of 1st category (by part (b)) #

Prop 4.8' Let (X, d) be a metric space.

(a) Every subset containing a residual set is residual.

(b) The intersection of countable many residual sets is a residual set.

(c) If (X, d) has no isolated point, then complement of a countable set is a residual set.

(Pf: By taking complement in Prop 4.8)

eg 4.5: $\mathbb{R}$ has no isolated point (in standard metric)

$\Rightarrow$ $\{q_j\}$ is nowhere dense $\forall$ rational number

$\Rightarrow$ $\mathbb{Q}$ is of 1st category

Hence $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ the set of irrational numbers is a residual set in $\mathbb{R}$.

Thm 4.9 (Baire Category Theorem)
In a complete metric space, any set of 1st category has empty interior.

Pf: Let the complete metric space be (X, d).

And let $E = \bigcup_{n=1}^{\infty} E_n \subset X$ be of 1st category

where E_n is nowhere dense in X, $\forall n$

Consider any open metric ball $B_{r_0}(x_0)$ of X.

Since $\overline{E_1}$ has empty interior (by defn. of nowhere denseness),

$$(X \setminus \overline{E_1}) \cap B_{r_0}(x_0) \neq \emptyset$$

Let $x_1 \in (X \setminus \overline{E_1}) \cap B_{r_0}(x_0)$.

Since both $X \setminus \overline{E_1}$ & $B_{r_0}(x_0)$ are open,

$\exists\ r_1 > 0$ s.t. $\overline{B_{r_1}(x_1)} \subset (X \setminus \overline{E_1}) \cap B_{r_0}(x_0)$

and $r_1 \leq \frac{r_0}{2}$ (as we can always choose a smaller ball)

$$\Rightarrow \quad \overline{B_{r_1}(x_1)} \cap \overline{E_1} = \emptyset$$

Now E_2 is nowhere dense, $\overline{E_2}$ has empty interior.

$$\Rightarrow \quad (X \setminus \overline{E_2}) \cap B_{r_1}(x_1) \neq \emptyset .$$

Similarly to the above, $\exists\, x_2 \in (X \setminus \overline{E_2}) \cap B_{r_1}(x_1)$ and $r_2 > 0$ with $r_2 \leq \frac{r_1}{2}$ such that

$$\overline{B_{r_2}(x_2)} \subset (X \setminus \overline{E_2}) \cap B_{r_1}(x_1) \quad \left(\begin{array}{l} \subset (X \setminus \overline{E_2}) \\ \subset B_{r_1}(x_1) \end{array} \right)$$

Note that $\overline{B_{r_2}(x_2)} \subset B_{r_1}(x_1) \subset (X \setminus \overline{E_1}) \cap B_{r_0}(x_0) \subset X \setminus \overline{E_1}$.

Repeating the process, we obtain $\{x_n\}_{n=1}^{\infty} \subset X$ and $\{r_n\}_{n=1}^{\infty} \subset \mathbb{R}_+$ such that

(a) $\overline{B_{r_{n+1}}(x_{n+1})} \subset B_{r_n}(x_n)$

(b) $r_{n+1} \leq \frac{r_n}{2}$

(c) $\overline{B_{r_n}(x_n)} \subset X \setminus \overline{E_j}, \ \forall j = 1, \cdots, n$

$\left(\overline{B_{r_n}(x_n)} \cap \overline{E_j} = \emptyset, \ \forall j = 1, \cdots, n \right)$

By (a) & (b), $\{x_n\}$ is a Cauchy seq. (Ex!)

Hence completeness of $X \Rightarrow \exists\, x \in X$ s.t. $x_n \to x$.

By (a) again, $x_{n+m} \in \overline{B_{r_n}(x_n)}, \ \forall m = 1, 2, 3, \cdots$

$$\Rightarrow \quad x \in \overline{B_{r_n}(x_n)}$$

By (a) & (c) $\quad x \in X \setminus \overline{E_n}$ and $B_{r_0}(x_0)$

Since n is arbitrary, $x \in \bigcap_{n=1}^{\infty} (X \setminus \overline{E_n}) = X \setminus \left(\bigcup_{n=1}^{\infty} \overline{E_n} \right)$

$$\Rightarrow \quad x \in \left(X \setminus \left(\bigcup_{n=1}^{\infty} \overline{E_n} \right) \right) \cap B_{r_0}(x_0)$$

$$\Rightarrow \left(X \setminus \left(\bigcup_{n=1}^{\infty} \overline{E_n} \right) \right) \cap B_{r_0}(x_0) \neq \emptyset$$

$$\Rightarrow \left(X \setminus \left(\bigcup_{n=1}^{\infty} E_n \right) \right) \cap B_{r_0}(x_0) \supset \left(X \setminus \left(\bigcup_{n=1}^{\infty} \overline{E_n} \right) \right) \cap B_{r_0}(x_0) \neq \emptyset$$

Since $B_{r_0}(x_0)$ is arbitrary, $E = \bigcup_{n=1}^{\infty} E_n$ has empty interior.

$\#$