§3.4 Picard–Lindelöf Theorem for Differential Equations

Let f be a function defined on

$$R = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b] \qquad \text{where } (t_0, x_0) \in \mathbb{R}^2 \text{ and } a, b > 0.$$

We consider Cauchy Problem (Initial Value Problem)

$$(\text{IVP}) \quad \begin{cases} \dfrac{dx}{dt} = f(t, x) \\ x(t_0) = x_0 \end{cases}$$

i.e. find a function $x(t)$ defined in a perhaps smaller interval

$$x : [t_0 - a', t_0 + a'] \to [x_0 - b, x_0 + b]$$

such that

$$\begin{cases} x(t) \text{ is differentiable,} \\ x(t_0) = x_0, \text{ and} \\ \dfrac{dx}{dt}(t) = f(t, x(t)), \ \forall t \in [t_0 - a', t_0 + a'] \end{cases}$$

for some $0 < a' \leq a$.

eg 3.14 Consider $\begin{cases} \dfrac{dx}{dt} = 1 + x^2 \\ x(0) = 0 \end{cases}$

Here $f(t, x) = 1 + x^2$ is smooth on $[-a, a] \times [-b, b]$ for any $a, b > 0$.

However, the solution $x(t) = \tan t$ defined only on $(-\frac{\pi}{2}, \frac{\pi}{2})$.

$\therefore$ Even for nice f, we may still have $a' < a$.

Recall :

(i) f defined in $R=[t_0-a, t_0+a]\times[x_0-b, x_0+b]$ satisfies the Lipschitz condition (uniform in t)

if $\exists L>0$ s.t. $\forall (t,x_1), (t,x_2)\in R$,

$$|f(t,x_1)-f(t,x_2)|\le L|x_1-x_2|.$$

(ii) In particular, $f(t,\cdot)$ is Lip. cts in x, $\forall t\in[t_0-a, t_0+a]$.

(iii) L is called a Lipschitz constant.

(iv) If L is a Lip. constant for f, then any $L'>L$ is also a Lip. constant.

(v) Not all cts. functions satisfy the Lip. condition.
eg. $f(t,x)=tx^{1/2}$ is cts, but not Lip. near 0.

(vi) If $R=[t_0-a, t_0+a]\times[x_0-b, x_0+b]$ and
$f(t,x): R\to\mathbb{R}$ is C^1,
then $f(t,x)$ satisfies the Lip. condition:
in fact, for some $y\in[x_0-b, x_0+b]$,

$$|f(t,x_2)-f(t,x_1)|=\left|\frac{\partial f}{\partial x}(t,y)(x_2-x_1)\right|$$

Hence $|f(t,x_2)-f(t,x_1)|\le L|x_2-x_1|$,

for $L=\max\left\{\left|\frac{\partial f}{\partial x}(t,x)\right| : (t,x)\in R\right\}$.

Thm 3.10 (Picard – Lindelöf Theorem)
(Fundamental Theorem of Existence and Uniqueness of Differential Equations)

Let f be continuous function on $R = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b]$, ($(t_0, x_0) \in \mathbb{R}^2$, $a, b > 0$) satisfies the Lipschitz condition on R (uniform in t). Then $\exists\ a' \in (0, a]$ and $x \in C^1[t_0 - a', t_0 + a']$ such that

$$x_0 - b \le x(t) \le x_0 + b, \quad \forall\ t \in [t_0 - a', t_0 + a']$$

and solving the Cauchy Problem (IVP)

Furthermore, x is the unique solution in $[t_0 - a', t_0 + a']$.

Note: One will see in the following proof that a' can be taken to be any number satisfying

$$0 < a' < \min\left\{a, \frac{b}{M}, \frac{1}{L}\right\}$$

where $M = \sup\{|f(t,x)| : (t,x) \in R\}$ &

L = Lip. const. for f.

Prop 3.11: Setting as in Thm 3.10, every solution x of (IVP) from $[t_0-a', t_0+a']$ to $[x_0-b, x_0+b]$ satisfies the equation

$$\boxed{x(t) = x_0 + \int_{t_0}^{t} f(t, x(t))\,dt} \quad (3.7)$$

Conversely, every $x(t) \in C[t_0-a', t_0+a']$ satisfying (3.7) is C^1 and solves (IVP).

Pf: Obvious, by Fundamental Theorem of Calculus.

Proof of Picard-Lindelöf Theorem:

For $a'>0$ to be chosen later, we let

$x_0-b \le \varphi(t) \le x_0+b$

$$\mathbb{X} = \{ \varphi \in C[t_0-a', t_0+a'] : \varphi(t_0) = x_0,\ \varphi(t) \in [x_0-b, x_0+b] \}$$

with (uniform) metric d_∞ on $\mathbb{X}$.

First note that $\mathbb{X}$ is a closed subset in the complete metric space $(C[t_0-a', t_0+a'], d_\infty)$. Hence $(\mathbb{X}, d_\infty)$ is complete.

Define T on $\mathbb{X}$ by

$$(T\varphi)(t) = x_0 + \int_{t_0}^{t} f(s, \varphi(s))\,ds$$

(This is well-defined as $\varphi(s) \in [x_0-b, x_0+b]$.)

Clearly $T\varphi \in C[t_0-a', t+a']$ & $(T\varphi)(t_0) = x_0$.

To show $T\varphi \in \mathbb{X}$, we need $(T\varphi)(t) \in [x_0-b, x_0+b]$.

Let $M = \sup_{(t,x)\in R} |f(t,x)|$.

Then $\forall\, t \in [t_0 - a', t_0 + a']$,

$$|(T\varphi)(t) - x_0| = \left|\int_{t_0}^{t} f(s,\varphi(s))\,ds\right| \leq M|t - t_0|$$

$$\leq Ma'$$

If we choose $0 < a' \leq \frac{b}{M}$, then

$$|(T\varphi)(t) - x_0| \leq b$$

$\Rightarrow \quad T\varphi \in X$.

This is, for $0 < a' \leq \frac{b}{M}$, $T : X \to X$ is a self-map from a complete metric space (X, d_∞) to itself.

To see whether T is a contraction, we check

$$|(T\varphi_2 - T\varphi_1)(t)| = \left|\left(x_0 + \int_{t_0}^{t} f(s,\varphi_2(s))ds\right) - \left(x_0 + \int_{t_0}^{t} f(s,\varphi_1(s))ds\right)\right|$$

$$\leq \int_{t_0}^{t} |f(s,\varphi_2(s)) - f(s,\varphi_1(s))|\,ds$$

$$\leq L\int_{t_0}^{t} |\varphi_2(s) - \varphi_1(s)|\,ds \quad \text{(by Lip. condition)}$$

$$\leq L|t - t_0| \sup_{[t_0-a',\, t_0+a']} |\varphi_2(s) - \varphi_1(s)|$$

$$\leq La'\, d_\infty(\varphi_2, \varphi_1)$$

Therefore, if we further require $La' = \gamma < 1$, then T is a contraction:

$$d_\infty(T\varphi_2, T\varphi_1) \le \gamma\, d_\infty(\varphi_2, \varphi_1) \quad \text{with } \gamma = La' < 1.$$

In conclusion, if $0 < a' < \min\{a, \frac{b}{M}, \frac{1}{L}\}$,

then $T: \mathbb{X} \to \mathbb{X}$ is a contraction on a complete metric space.

Therefore, by Contraction Mapping Principle, T admits a unique fixed point $x(t) \in \mathbb{X}$.

By Prop 3.11, we've proved Thm 3.10. ※

Notes:

(1) Existence part of Picard–Lindelöf Thm still holds with $f(t,x)$ cts only (without Lip. condition). However, the solution may not be unique:

eg: Consider $f(t,x) = |x|^{1/2}$ on $\mathbb{R}\times\mathbb{R}$. f is cts, but not Lip. cts.

(Cauchy Problem) $\begin{cases} \frac{dx}{dt} = |x|^{1/2} & \text{on } \mathbb{R} \\ x(0) = 0 \end{cases}$

has solutions $x_1(t) \equiv 0$ and $x_2(t) = \begin{cases} \frac{1}{4}t^2, & t \ge 0 \\ -\frac{1}{4}t^2, & t < 0 \end{cases}$

(Check: x_2 is differentiable with $\frac{dx_2}{dt} = \frac{1}{2}|t| = |x_2|^{1/2}$, $\forall t \in \mathbb{R}$
(& $x_2(0) = 0$))

(2) Uniqueness holds regardless of the size of the interval of existence. (Proof omitted as it is more in the curriculum of ODE. See Prof Chou's notes for a proof.)

(3) The proof works for system of ODEs, just the x and f become vector-valued:

Thm 3.13 (Picard-Lindelöf Theorem for Systems)

Consider (IVP) $\begin{cases} \dfrac{dx}{dt} = f(t,x) \\ x(t_0) = x_0, \end{cases}$ $\quad x_0 = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

where $x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} \in [x_1-b, x_1+b] \times \cdots \times [x_n-b, x_n+b]$ and

$$f(t,x) = \begin{pmatrix} f_1(t,x) \\ \vdots \\ f_n(t,x) \end{pmatrix} \in C^1(R), \text{ with}$$

$$R = [t_0-a, t_0+a] \times [x_1-b, x_1+b] \times \cdots \times [x_n-b, x_n+b],$$

satisfying (Lipschitz condition (uniform in t))

$$|f(t,x) - f(t,y)| \le L|x-y|, \quad \forall (t,x), (t,y) \in R,$$

for some constant $L > 0$.

There exists a unique solution $x \in C^1[t_0-a', t_0+a']$ with

$$x(t) \in [x_1-b, x_1+b] \times \cdots \times [x_n-b, x_n+b], \quad \forall t \in [t_0-a', t_0+a']$$

to (IVP), where a' satisfies

$$0 < a' < \min\left\{a, \frac{b}{M}, \frac{1}{L}\right\}, \quad \text{here } M = \max_{j=1,\cdots,n} \sup_R |f_j(t,x)|.$$

(4) The Picard-Lindelöf Theorem for system can be applied to initial value problem for higher order ordinary differential equations:

$$(IVP)\begin{cases} \dfrac{d^m x}{dt^m} = f\left(t, x, \dfrac{dx}{dt}, \cdots, \dfrac{d^{m-1}x}{dt^{m-1}}\right) \\ x(t_0) = x_0 \\ \dfrac{dx}{dt}(t_0) = x_1 \\ \vdots \\ \dfrac{d^{m-1}x}{dt^{m-1}}(t_0) = x_{m-1} \end{cases}$$

By letting $\vec{X} = \begin{pmatrix} x \\ \frac{dx}{dt} \\ \vdots \\ \frac{d^{m-1}x}{dt^{m-1}} \end{pmatrix}$, then

$$\frac{d\vec{X}}{dt} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{d^2x}{dt^2} \\ \vdots \\ \frac{d^m x}{dt^m} \end{pmatrix} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{d^2x}{dt^2} \\ \vdots \\ f\left(t, x, \frac{dx}{dt}, \cdots, \frac{d^{m-1}x}{dt^{m-1}}\right) \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{pmatrix} = \vec{f}(t, \vec{X})$$

with $\vec{X}(t_0) = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{m-1} \end{pmatrix}$.

Ch 4 Space of Continuous Functions

§4.1 Ascoli's Theorem

Notation: If (X, d) = metric space, we denote

$$C_b(X) = \{ f \in C(X) : |f(x)| \leq M, \forall x \in X, \text{ for some } M \}$$

the vector space of all bounded continuous functions on X.
Clearly $C_b(X) \subset C(X)$.
($C(X)$ = set of continuous functions on X.)

eg: If G = (non empty) bounded open set in $\mathbb{R}^n$, then

$$C_b(\bar{G}) = C(\bar{G})$$

as $\bar{G}$ is closed and bounded, $f \in C(\bar{G})$ has to be bounded.

Recall: A norm $\|\cdot\|$ on a real vector space X is defined by the following properties:

(N1) $\|x\| \geq 0$ & "$\|x\| = 0 \Leftrightarrow x = 0$"

(N2) $\|\alpha x\| = |\alpha| \|x\|$ $(\alpha \in \mathbb{R})$

(N3) $\|x+y\| \leq \|x\| + \|y\|$.

And a vector space with norm $(X, \|\cdot\|)$ is called a norm space. A norm space has a natural metric

$$d(x,y) = \|x - y\|.$$

Fact: The <u>supnorm</u> $\|f\|_\infty = \sup_{x\in X} |f(x)|$ is a norm on $C_b(X)$.

And we always assume $C_b(X)$ with metric

$$d_\infty(f,g) = \|f-g\|_\infty .$$

given by the <u>supnorm</u>.

Similar to $(C[a,b], d_\infty)$, we have

$\boxed{\text{Prop: } (C_b(X), d_\infty) \text{ is complete. (for any metric space } (X,d))}$

Pf: Let $\{f_n\}$ be a Cauchy seq. in $(C_b(X), d_\infty)$.
Then $\forall \varepsilon>0$, $\exists n_0 \geq 0$ s.t.

$$\|f_m - f_n\|_\infty < \frac{\varepsilon}{4}, \quad \forall m,n \geq n_0 .$$

In particular, $\forall x\in X$,

$$(*)_1 \quad |f_m(x) - f_n(x)| \leq \|f_m - f_n\|_\infty < \frac{\varepsilon}{4}, \ \forall m,n \geq n_0$$

$\Rightarrow$ $\{f_n(x)\}$ is a Cauchy seq. in $\mathbb{R}$.

By <u>completeness</u> of $\mathbb{R}$ (not X), $\lim_{n\to\infty} f_n(x)$ exists and, in general, depends on x. Let denote it by

$$f(x) = \lim_{n\to\infty} f_n(x), \quad \forall x\in X.$$

This gives a function f on X.

Claim 1 f is bounded.

Pf: Letting $m \to \infty$ in $(*)_1$, we have $\forall \varepsilon > 0$, and $\forall x \in X$,

$$(*)_2 \quad |f(x) - f_n(x)| \leq \frac{\varepsilon}{4}, \quad \forall n \geq n_0$$

In particular, $|f(x) - f_{n_0}(x)| \leq \frac{\varepsilon}{4}$, $\forall \varepsilon > 0$, $\forall x \in X$.

$\Rightarrow \forall x \in X$, $|f(x)| \leq \frac{\varepsilon}{4} + |f_{n_0}(x)| \leq \frac{\varepsilon}{4} + M_0$,

where M_0 is a bound for f_{n_0}.

$\therefore$ f is bounded.

Claim 2 : f is continuous.

Pf : f_{n_0} cts $\Rightarrow \forall x_0 \in X$ & $\forall \varepsilon > 0$, $\exists \delta > 0$

s.t. $|f_{n_0}(x) - f_{n_0}(x_0)| < \frac{\varepsilon}{4}$, $\forall d(x, x_0) < \delta$.

Then together with $(*)_2$,

$$|f(x) - f(x_0)| \leq |f(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(x_0)| + |f_{n_0}(x_0) - f(x_0)|$$

$$\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon, \qquad \forall d(x, x_0) < \delta.$$

$\therefore$ f is cts at x_0.

Since $x_0 \in X$ is arbitrary, f is cts on X.

Claims 1 & 2 $\Rightarrow f \in C_b(X)$.

Finally, by $(*)_2$, $\sup_{x \in X} |f(x) - f_n(x)| \leq \frac{\varepsilon}{4}$, $\forall n \geq n_0$

i.e. $d_\infty(f_n, f) \le \frac{\varepsilon}{4}, \quad \forall n \ge n_0$

So $d_\infty(f_n, f) \to 0$ as $n \to \infty$

That is $f_n \to f$ in $(C_b(X), d_\infty)$. ✗

<u>Notes</u>: (i) We've just proved that $(C_b(X), d_\infty)$ is a <u>Banach space</u>, i.e. a <u>complete</u> normed vector space.

(ii) $C_b(X)$ is <u>usually</u> of <u>infinite dimensional</u>:

<u>egs</u>: When $X = \mathbb{R}^n$ or subset with non-empty interior in $\mathbb{R}^n$.

Explicit eg: $X = [0,1] \subset \mathbb{R}$, then

$$\{f_n(x) = x^n\}_{n=0}^{\infty} \subset C_b(X).$$

Clearly, $\{x^n\}_{n=0}^{\infty}$ is a linearly indep. subset.

$\Rightarrow C_b(X) = C[0,1]$ is of infinite dimensional.

(iii) $C_b(X)$ could be of <u>finite dimension</u>:

eg: $X = \{p_1, \cdots, p_n\}$ finite set with discrete metric

Then
$$\begin{array}{ccc} X & \to & \mathbb{R}^n \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ f & \mapsto & (f(p_1), \cdots, f(p_n)) \end{array}$$
is a linear bijection.

(iv) A reason for studying $C_b(X)$ instead of $C(X)$ is the fact that $C(X)$ may contain unbounded function and supnam $\|\cdot\|_\infty$ doesn't define.

$$eg: \ X = \mathbb{R} = (-\infty, +\infty).$$

However, in some cases, it is still possible to define a metrix on $C(X)$.

eg $X = \mathbb{R}^m$, $\overline{B_n(0)} = \{x \in \mathbb{R}^m : |x| \leq n\}$, $\forall n = 1, 2, 3, \cdots$

$\forall f \in C(\mathbb{R}^m)$, define

$$d(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|f-g\|_{\infty, \overline{B_n(0)}}}{1 + \|f-g\|_{\infty, \overline{B_n(0)}}}$$

where $\|\cdot\|_{\infty, \overline{B_n(0)}}$ is the supnorm on the closed ball $\overline{B_n(0)}$.

Then d is a complete metric on $C(\mathbb{R}^n)$. (Ex!)

(v) $C_b(X)$ may <u>not</u> have Bolzano-Weierstrass property.

<u>Recall</u>:

Bolzano-Weierstrass Theorem (in $\mathbb{R}^n$):   Every bounded sequence (set) has (contains) a convergent subsequence (sequence).

eg. $C_b([0,1]) = C[0,1]$. Let $f_n(x) = x^n$, $x \in [0,1]$.

Then $\|f_n\|_\infty = 1$, $\forall n$.

Note that pointwise limit $f_n(x) \to \begin{cases} 1, & x=1 \\ 0, & \text{otherwise.} \end{cases}$

$\Rightarrow$ no subsequence converges in $C_b([0,1])$.

In view of note (V), we need further condition to help us to find convergence sequence in subset of $C_b(X)$.

Def: Let (X, d) be a metric space. A set $E \subset X$ is called a <u>precompact</u> set if every sequence in E contains a <u>convergent subsequence</u> (with limit in X, not necessary in E). If further required that the limit belongs to E, then it is called <u>compact</u>.

<u>Note</u>: Compact set is a closed precompact set.

Pf: Let $\{x_n\} \subset E$. E precompact $\Rightarrow \exists\, x_{n_j} \to z \in X$

E closed $\Rightarrow z \in E$

Hence closed precompact $\Rightarrow$ compact.

The other direction: "compact $\Rightarrow$ closed precompact" is trivial. #

eg: Bolzano-Weierstrass $\Rightarrow$

$E \subset \mathbb{R}^n$ is precompact $\Leftrightarrow$ E is bounded.

Hence $E \subset \mathbb{R}^n$ is compact $\Leftrightarrow$ E is closed & bounded.

<u>Def</u>: Let (X, d) be a metric space. A subset $\mathcal{C}$ of $C(X)$ is <u>equicontinuous</u> if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon, \ \forall f \in \mathcal{C} \ \& \ d(x,y) < \delta \ (x, y \in X).$$

<u>Note</u>: Clearly if $\mathcal{C}$ is equicontinuous, then any $\mathcal{C}' \subset \mathcal{C}$ is equicontinuous.

<u>Eg</u>: If $X = \overline{G} \subset \mathbb{R}^n$, $G \neq \emptyset$ open & bounded. Then $f \in C(\overline{G})$ is always uniformly continuous:

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$|f(x) - f(y)| < \varepsilon, \ \forall \ d_{\mathbb{R}^n}(x,y) = |x-y| < \delta \ (x, y \in \overline{G})$$

The δ here usually depends on f. Comparing to the definition of equicontinuity, $\mathcal{C}$ is equicontinuous, if we can find a <u>uniform $\delta > 0$ for all functions $f \in \mathcal{C}$</u>, i.e. δ is <u>independent</u> of points $x, y \in \overline{G}$ and functions $f \in \mathcal{C}$.

eg: A function f defined on a subset $\bar{G}$ of $\mathbb{R}^n$ (non-empty open & bound G) is called Hölder continuous if $\exists\ \alpha \in (0,1)$ such that

$$(*)\quad |f(x)-f(y)| \leq L|x-y|^{\alpha}, \quad \forall x,y \in \bar{G},$$

for some constant L.

The number α is called the Hölder exponent.

The function is called Lipschitz continuous if $(*)$ holds for $\alpha = 1$.

For a fixed $\alpha \in (0,1]$ & $L>0$, the family

$$\mathcal{C} = \{ f \in C(\bar{G}) : f \text{ Hölder/Lip. with exponent } \alpha \text{ and } L>0 \}$$

is an equicontinuous family.

Pf: $\forall \varepsilon > 0$, let $\delta > 0$ such that $L\delta^{\alpha} < \varepsilon$.

Then $\forall f \in \mathcal{C}$, $\forall x,y \in X$ with $|x-y|<\delta$,

$$|f(x)-f(y)| \leq L|x-y|^{\alpha} < L\delta^{\alpha} < \varepsilon. \quad \#$$

Prop 4.1: Let $\mathcal{C}$ be a subset $C(\bar{G})$ where $\bar{G}$ is a nonempty convex in $\mathbb{R}^n$ (with G open & bounded). Suppose that each function in $\mathcal{C}$ is differentiable and there is a uniform bound on their partial derivatives. Then $\mathcal{C}$ is equicontinuous.

(i.e. $\mathcal{C} = \{ f \in C(\bar{G}) : f \text{ differentiable}, \left\| \frac{\partial f}{\partial x_i} \right\|_{\infty} \leq M, \forall i \}$ is equicontinuous provided $\bar{G}$ is convex. (for same M))

Pf: $\forall\, x,y \in \overline{G}$, $\overline{G}$ convex

$$\Rightarrow \quad x+t(y-x) \in \overline{G}, \quad \forall\, t \in [0,1].$$

Then $f(y)-f(x) = \int_0^1 \frac{d}{dt} f(x+t(y-x))\,dt$

$$= \int_0^1 \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x+t(y-x))\,(y_i-x_i)\,dt$$

$$= \sum_{i=1}^{n} \left(\int_0^1 \frac{\partial f}{\partial x_i}(x+t(y-x))\,dt \right)(y_i-x_i)$$

$$\leq \sqrt{\sum_{i=1}^{n} \left| \int_0^1 \frac{\partial f}{\partial x_i}(x+t(y-x))\,dt \right|^2}\ |y-x|$$

$\leq \sqrt{n}\, M\, |y-x|$, where M = uniform bd. on the partial derivatives

Interchanging x, y, we have

$$|f(y)-f(x)| \leq \sqrt{n}\, M\, |y-x|, \quad \forall\, x,y \in \overline{G}$$

Then by the above example, $\mathcal{C}$ is equicontinuous. ※