Final Step for special case: G is $C^1(B_{\frac{r}{2}}(0))$

furthermore, if F is C^k $(k \geq 1)$, then G is C^k.

Pf of final step: By assumption, DF is continuous and invertible on $B_r(0)$. Linear Algebra $\Rightarrow$ $(DF)^{-1}$ is continuous.

Therefore, by Step 3 (and step 2) $DG(y) = (DF)^{-1}(G(y))$ is continuous. Hence G is C^1.

The fact that F is C^k $(k \geq 1)$ $\Rightarrow$ G is C^k is by differentiating the identity $DG(y) = (DF)^{-1}(G(y))$ and using induction. $\divideontimes$

General Case:

Consider $\tilde{F}(x) = (DF)^{-1}(x_0)\left[F(x+x_0) - y_0\right]$

Then $\tilde{F}(0) = 0$

$\tilde{F}$ defined on an open set $\tilde{U} = U - x_0 = \{x : x + x_0 \in U\}$ with $0 \in \tilde{U}$.

and $D\tilde{F}(0) = (DF)^{-1}(x_0)\, DF(x_0) = I$.

By the special case, $\exists\, r > 0$ such that

$\exists\ \tilde{G} : B_{\frac{r}{2}}(0) \longrightarrow \tilde{G}(B_{\frac{r}{2}}(0)) \subset B_r(0) \subset \tilde{U}$

s.t. $\tilde{G}$ is the local inverse of $\tilde{F}$.

Let $W = DF(x_0)(B_{\frac{r}{2}}(0)) + y_0$

$V = \tilde{G}(B_{\frac{r}{2}}(0)) + x_0$, (then $V \subset U$ & $x_0 \in V$)

and $G: W \to V$ by

$$G(y) = \tilde{G}\left((DF)^{-1}(x_0)(y - y_0)\right) + x_0, \quad \forall y \in W.$$

Clearly G maps W bijective onto V.

Since $\tilde{F}(x) = (DF)^{-1}(x_0)\left[F(x + x_0) - y_0\right]$,

we have $F(x + x_0) = (DF)(x_0)\tilde{F}(x) + y_0, \quad \forall x \in B_r(0)$

$\Rightarrow \quad F(x) = (DF)(x_0)\tilde{F}(x - x_0) + y_0, \quad x \in V$

Hence $\forall y \in W$

$$
\begin{aligned}
F(G(y)) &= (DF)(x_0)\tilde{F}(G(y) - x_0) + y_0 \\
&= y_0 + (DF)(x_0)\tilde{F}\left[\tilde{G}\left((DF)^{-1}(x_0)(y - y_0)\right)\right] \quad \left(\text{by definition of } G\right) \\
&= y_0 + (DF)(x_0)\left((DF)^{-1}(x_0)(y - y_0)\right) \quad \left(\text{since } \tilde{F} \circ \tilde{G} = I\right) \\
&= y_0 + y - y_0 = y
\end{aligned}
$$

$\therefore$ G is the local inverse of F

The remaining facts that $F \in C^k$ $(k \geq 1) \Rightarrow G \in C^k$, is clear from the definition of G, and the results on $\tilde{G}$ (& $\tilde{F}$) in the special case. ###

Def: A C^k-map $F: V \to W$ (V, W open in $\mathbb{R}^n$) is a <u>C^k-diffeomorphism</u> if F^{-1} exists and is also C^k.

<u>Note</u>: (i) The IFT can be rephased as:

If $F: U \overset{\subset \mathbb{R}^n}{\longrightarrow} \mathbb{R}^n \in \underline{C^k}$, and DF is <u>nonsingular</u>

at a point $x_0 \in U$, then F is a C^k-diffeomorphism between some nbds V and W of x_0 & $F(x_0)$ respectively.

(ii) If $F: V \to W$ is a C^k-diffeomorphism, then $\forall$ function $\varphi: W \to \mathbb{R}$, there corresponds a function

$$\psi = \varphi \circ F : V \to \mathbb{R}.$$

Conversely, $\forall$ function $\psi: V \to \mathbb{R}$, there corresponds a function

$$\varphi = \psi \circ F^{-1} : W \to \mathbb{R}.$$

Moreover, $\underline{\varphi \text{ is } C^k \Leftrightarrow \psi \text{ is } C^k}$.

Thus every C^k-diffeomaphism gives rise to a <u>"local C^k-change of coordinates"</u>.

Thm 3.5 (Implicit Function Theorem)

Let U be an open set in $\mathbb{R}^n \times \mathbb{R}^m$

$F : U \to \mathbb{R}^m$ is a C^1-map.

Suppose that $(x_0, y_0) \in U$ satisfies $F(x_0, y_0) = 0$, and $D_y F(x_0, y_0)$ is invertible in $\mathbb{R}^m$. Then

(1) $\exists$ an open set of the form $V_1 \times V_2 \subset U$ containing (x_0, y_0) and a C^1-map

$$\varphi : \overset{\subset \mathbb{R}^n}{V_1} \longrightarrow \overset{\subset \mathbb{R}^m}{V_2} \quad \text{with } \varphi(x_0) = y_0$$

such that $F(x, \varphi(x)) = 0, \ \forall x \in V_1$.

(2) $\varphi : V_1 \to V_2$ is C^k when F is C^k, $1 \leq k \leq \infty$.

(3) Moreover, assume further that DF_y is invertible in $V_1 \times V_2$. Then, if $\psi : V_1 \to V_2$ is another C^1-map satisfying

$F(x, \psi(x)) = 0$, we have $\psi \equiv \varphi$.

Note: If $F = \begin{pmatrix} F_1(x_1, \dots, x_n, y_1, \dots, y_m) \\ \vdots \\ F_m(x_1, \dots, x_n, y_1, \dots, y_m) \end{pmatrix}$, then

$$D_yF = \begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \vdots & & \vdots \\ \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial y_m} \end{pmatrix}$$

is $m \times m$ & can be regarded as a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^m$

In general, for a map F such that $DF(x_0,y_0)$ has <u>rank m</u>, then one can rearrange the independent variables to make the $m\times m$ submatrix corresponding to the <u>last m columns</u> of the Jacobian matrix <u>invertible</u>, i.e. in the situation of the theorem.

Hence the condition that <u>$DF_y(x_0,y_0)$ is invertible</u> in the Implicit Function Theorem can be generalized to <u>$\text{rank}\, DF(x_0,y_0) = m$</u>.

<u>Pf of Implicit Function Theorem</u>: (using Inverse Function Theorem)

Define
$$\begin{array}{rcl} \Phi : U \subset \mathbb{R}^n \times \mathbb{R}^m & \longrightarrow & \mathbb{R}^n \times \mathbb{R}^m \\ \psi & & \psi \\ (x,y) & \longmapsto & (x, F(x,y)) \end{array}$$

where $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$, $y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} \in \mathbb{R}^m$.

Then $\Phi(x_0,y_0) = (x_0, 0)$.

Clearly Φ is C^k if F is C^k.

And

$$\Phi = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ F_1(x_1,\cdots,x_n,y_1,\cdots,y_m) \\ \vdots \\ F_m(x_1,\cdots,x_n,y_1,\cdots,y_m) \end{pmatrix}$$

$$\Rightarrow \quad D\Phi = \left(\begin{array}{ccc|ccc} 1 & & 0 & & & \\ & \ddots & & & 0 & \\ 0 & & 1 & & & \\ \hline \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} & \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} & \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial y_m} \end{array}\right)$$

Since $D_yF\Big|_{(x_0,y_0)} = \begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \vdots & & \vdots \\ \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial y_m} \end{pmatrix}$ is invertible in $\mathbb{R}^m$,

$D\Phi\Big|_{(x_0,y_0)}$ is invertible in $\mathbb{R}^n \times \mathbb{R}^m$.

Applying Inverse Function Theorem, $\exists$ local C^1-inverse

$$\Psi = (\Psi_1, \Psi_2) : W^{\subset \mathbb{R}^n \times \mathbb{R}^m} \longrightarrow V^{\subset U},$$

with $(x_0, y_0) = \Psi(x_0, 0) = (\Psi_1(x_0,0), \Psi_2(x_0,0))$,

where W and V are open nbds. of $\Phi(x_0,y_0) = (x_0,0)$ and (x_0,y_0) respectively, and is C^k when F is C^k.

By shrinking the nbds, we may assume V is of the form $V_1 \times V_2$, where V_1 open in $\mathbb{R}^n$ containing x_0; V_2 open in $\mathbb{R}^m$ containing y_0.

Now $\forall\ (x,z) \in W$,

$$\begin{aligned} (x,z) &= \Phi\left(\Psi_1(x,z),\ \Psi_2(x,z)\right) \\ &= \left(\Psi_1(x,z),\ F(\Psi_1(x,z), \Psi_2(x,z))\right) \end{aligned}$$

$$\therefore \begin{cases} x = \Psi_1(x,z) \\ z = F(\Psi_1(x,z), \Psi_2(x,z)) \end{cases}$$

$$\Rightarrow \quad z = F(x, \Psi_2(x,z))$$

In particular, we can take $z=0$ & hence

$$F(x, \Psi_2(x,0)) = 0, \quad \forall\, x = \Psi_1(x,0) \in V_1.$$

$\therefore\ \varphi: V_1 \to V_2 : x \mapsto \Psi_2(x,0)$ is the required map

s.t. $\begin{cases} \varphi(x_0) = \Psi_2(x_0,0) = y_0, \\ F(x, \varphi(x)) = 0 \end{cases}$

and is C^k when F is C^k. We've proved (1) & (2).

For (3), $D_y F$ is invertible in $V_1 \times V_2$

$$\Rightarrow \int_0^1 D_y F(x, y_1 + t(y_2 - y_1))\, dt \text{ is nonsingular}$$

for (x, y_1) & $(x, y_2) \in V_1 \times V_2$. (May assume V_2 is a ball)

Now if $\psi: V_1 \to V_2$ is another C^1-map s.t.

$$F(x, \psi(x)) = 0,$$

then
$$\begin{aligned} 0 &= F(x,\psi(x)) - F(x,\varphi(x)) \\ &= \left(\int_0^1 D_y F(x, \varphi(x) + t(\psi(x) - \varphi(x)))\, dt\right)(\psi(x) - \varphi(x)) \end{aligned}$$

$\int_0^1 D_y F(x, \varphi(x) + t(\psi(x) - \varphi(x)))\, dt$ nonsingular $\Rightarrow$

$$\psi(x) \equiv \varphi(x), \quad \forall\, x \in V_1. \quad \#$$

Remark: Implicit Function Theorem and Inverse Function Theorem are in fact equivalent:

If $F: U \to \mathbb{R}^n$ as in assumption of the Inverse Function Theorem, then define $\tilde{F}(x,y): U \times \mathbb{R}^n \to \mathbb{R}^n$ (n+n to n-dim)

$$(x, y) \mapsto F(x) - y$$

(i.e. $m=n$ in the theorem)

which is C^1.

Note that $\tilde{F}(x_0, y_0) = F(x_0) - y_0 = 0$, and

$$D_x \tilde{F}(x_0, y_0) = DF(x_0) \text{ is invertible}$$

$\Rightarrow$ $D\tilde{F}(x_0, y_0)$ is of full rank (rank $D\tilde{F}(x_0, y_0) = n$)

By Implicit Function Theorem, $\exists$ C^1-mapping $\varphi(y)$ near y_0 such that

$$\varphi(y_0) = x_0 \quad \text{and} \quad \tilde{F}(\varphi(y), y) = 0.$$

(Note the different in the notations)

i.e. $F(\varphi(y)) - y = 0$ near y_0

$\therefore$ $x = \varphi(y)$ is the local inverse.

[Concrete examples are omitted since it should be given in advanced calculus already. A few explicit examples are given in Prof Chou's notes.]