Def: If a normed space $(\mathbb{X}, \|\cdot\|)$ is complete as a metric space with respect to the induced metric $d(x,y) = \|x-y\|$, $\forall x,y \in \mathbb{X}$. Then it is called a Banach space.

eg. - $(\mathbb{R}^n, \|\cdot\|_p)$ $(p \geq 1)$ is a Banach space.
- $(C[a,b], \|\cdot\|_\infty)$ is a Banach space.

Thm 3.4 (Perturbation of Identity)

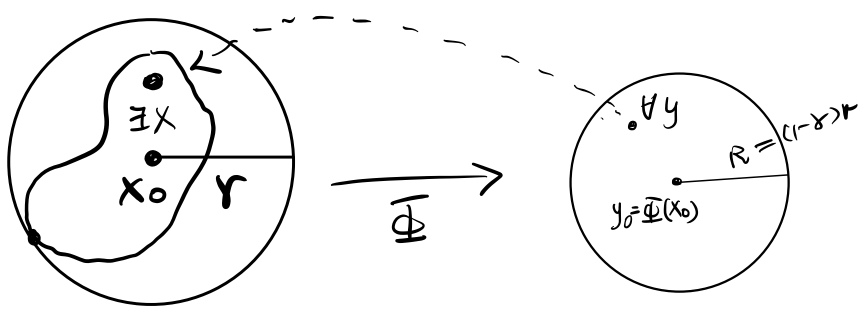
Let $(\mathbb{X}, \|\cdot\|)$ be a Banach space, and

$$\Phi : \overline{B_r(x_0)} \to \mathbb{X} \text{ satisfies } \Phi(x_0) = y_0$$

Suppose that $\Phi = Id_{\mathbb{X}} + \Psi$ such that $\exists$ constant $\gamma \in (0,1)$ such that

$$\|\Psi(x_2) - \Psi(x_1)\| \leq \gamma \|x_2 - x_1\|, \quad \forall x_1, x_2 \in \overline{B_r(x_0)}.$$

Then $\forall y \in \overline{B_R(y_0)}$, where $R = (1-\gamma) r$, $\exists$ unique $x \in \overline{B_r(x_0)}$ such that $\Phi(x) = y$.



(i.e. Φ is locally invertible)

Idea of proof:

$$y = \Phi(x) = (Id_X + \Psi)(x) = x + \Psi(x)$$

$$\Leftrightarrow \quad x = y - \Psi(x)$$

If $\forall y \in \overline{B_R(x_0)}$, define $Tx = y - \Psi(x)$.

Then $y = \Phi(x) \Leftrightarrow Tx = x$ (i.e. x is a fixed point of T).

<u>Proof</u>: Define $\tilde{\Phi} : \overline{B_r(0)} \to X$ by

$$\tilde{\Phi}(x) = \Phi(x+x_0) - \Phi(x_0)$$

$$= \left(x + x_0 + \Psi(x+x_0)\right) - \left(x_0 + \Psi(x_0)\right)$$

$$= x + \left[\Psi(x+x_0) - \Psi(x_0)\right] = x + \tilde{\Psi}(x)$$

Then $\tilde{\Phi}(0) = 0$.

Further define, for any $y \in \overline{B_R(0)}$ $(R = (1-\gamma)r)$ the map

$$T : \overline{B_r(0)} \to X \quad \text{by} \quad Tx = y - \tilde{\Psi}(x)$$

Then $\forall x \in \overline{B_r(0)}$,

$$\|Tx\| = \|y - \tilde{\Psi}(x)\| \le \|y\| + \|\Psi(x+x_0) - \Psi(x_0)\|$$

$$\le \|y\| + \gamma\|x\| \le R + \gamma r = r$$

$$\therefore \quad T : \overline{B_r(0)} \to \overline{B_r(0)}$$

And $\forall x_1, x_2 \in \overline{B_r(0)}$,

$$\|Tx_1 - Tx_2\| = \|[y - (\Psi(x_1+x_0) - \Psi(x_0))] - [y - (\Psi(x_2+x_0) - \Psi(x_0))]\|$$

$$= \|\Psi(x_1+x_0) - \Psi(x_2+x_0)\|$$

$$\leq \gamma \|x_1 - x_2\|$$

Since $\gamma \in (0,1)$, $T: \overline{B_r(0)} \to \overline{B_r(0)}$ is a contraction.

Since $\overline{B_r(0)}$ is a closed subset and $(X, \|\cdot\|)$ is complete, Prop 3.1 $\Rightarrow$ $\overline{B_r(0)}$ is also complete.

Hence one can apply Contraction Mapping Principle to conclude that $\exists$ unique $x \in \overline{B_r(0)}$ s.t.

$$Tx = x \quad \text{in } \overline{B_r(0)}.$$

i.e. $\quad x = y - (\Psi(x+x_0) - \Psi(x_0))$

$$= y - [(\Phi(x+x_0) - (x+x_0)) - (\Phi(x_0) - x_0)]$$

$$= y - \Phi(x+x_0) + \Phi(x_0) + x$$

$$\Leftrightarrow \quad \Phi(x+x_0) = y + y_0 \qquad (y_0 = \Phi(x_0))$$

Note that $y + y_0 \in \overline{B_R(y_0)}$ is arbitrary, and $x + x_0 \in \overline{B_r(x_0)}$, we've proved the Thm. $\#$

Remarks

(1) Only need to assume Φ (and Ψ) defined on $B_r(x_0)$ (open ball) satisfying $\|\Psi(x_1)-\Psi(x_2)\| \le \gamma \|x_1-x_2\|$, $\gamma\in(0,1)$ for $x_1, x_2 \in B_r(x_0)$ (open ball). Then it is easy to extend Φ (and Ψ) to $\overline{B_r(x_0)}$ and get the same inequality for all $x_1, x_2 \in \overline{B_r(x_0)}$.

(2) Actually one can prove more that if $y\in B_R(y_0)$ (open ball), then the solution $x \in B_r(x_0)$ (open ball).
(check the details of the pf.)

(3) The Thm $\Rightarrow$ $\Phi^{-1}: \overline{B_R(y_0)} \to \overline{B_r(x_0)}$ exists.

Claim $\quad \|\Phi^{-1}(y_1)-\Phi^{-1}(y_2)\| \le \frac{1}{1-\gamma}\|y_1-y_2\|, \ \forall y_1,y_2\in\overline{B_R(y_0)}$.

In particular, Φ^{-1} is uniformly continuous (in fact "Lipschitz"). (see Homework 5)

Pf: Let $x_i = \Phi^{-1}(y_i)$. Then x_i is the fixed point such that $\quad x_i = y_i - \Psi(x_i)$.

$$\begin{aligned}
\Rightarrow \|\Phi^{-1}(y_1)-\Phi^{-1}(y_2)\| &= \|(y_1-\Psi(x_1)) - (y_2-\Psi(x_2))\| \\
&\le \|y_1-y_2\| + \|\Psi(x_1)-\Psi(x_2)\| \\
&\le \|y_1-y_2\| + \gamma\|x_1-x_2\| \\
&= \|y_1-y_2\| + \gamma\|\Phi^{-1}(y_1)-\Phi^{-1}(y_2)\|
\end{aligned}$$

$$\Rightarrow \|\Phi^{-1}(y_1)-\Phi^{-1}(y_2)\| \le \frac{1}{1-\gamma}\|y_1-y_2\| \quad \#$$

eg 3.6: $3x^4 - x^2 + x = \underbrace{-0.05}_{\text{small}}$ has a real root.

(Observation: and $3x^4 - x^2 + x = 0$ has a root $x=0$.

Idea: look for solution near $x=0$ using Thm 3.4.)

Pf: Let $\Phi(x) = x + (3x^4 - x^2) = x + \Psi(x)$,

where $\Psi(x) = 3x^4 - x^2$ ("small" near $x=0$)

Then $\Phi(0) = 0$.

And for $x_1, x_2 \in \overline{B_r(0)}$ ($r>0$ to be determined)

$$
\begin{aligned}
|\Psi(x_1) - \Psi(x_2)| &= |(3x_1^4 - x_1^2) - (3x_2^4 - x_2^2)| \\
&= |3(x_1^4 - x_2^4) - (x_1^2 - x_2^2)| \\
&= |3(x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3) - (x_1 + x_2)|\,|x_1 - x_2| \\
&\leq (12r^3 + 2r)\,|x_2 - x_1|.
\end{aligned}
$$

Hence, we need to choose $r>0$ small enough such that $\gamma = 12r^3 + 2r < 1$

Also, in order to include $-0.05 \in \overline{B_R(0)}$, we need $R = (1-\gamma)r \geq 0.05$.

A choice is $r = \frac{1}{4}$.

Then $\gamma = \frac{11}{16} < 1$ and $R = (1-\gamma)r = \frac{5}{64} \sim 0.078$.

By Thm 3.4, $\forall\, y \in \overline{B_{\frac{5}{64}}(0)}$, $\exists\, x \in \overline{B_{\frac{1}{4}}(0)}$ s.t. $\Phi(x) = y$

i.e. $x + 3x^4 - x^2 = y$.

In particular, $-0.05 \in \overline{B_{\frac{5}{64}}(0)}$, we has a root of

$$x + 3x^4 - x^2 = -0.05. \quad \#$$

One can generalize eg 3.6 to

Prop 3.5: Let $\Phi(x) = x + \Psi(x) : U \to \mathbb{R}^n$ be C^1 on some open set $U \subset \mathbb{R}^n$ containing 0, such that

$$\Psi(0) = 0 \quad \text{and} \quad \lim_{x \to 0} \frac{\partial \Psi_i}{\partial x_j}(x) = 0, \ \forall i,j.$$

Then $\exists\ r > 0$ and $R > 0$ such that $\forall y \in B_R(0)$,

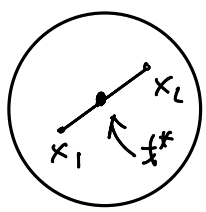
$\Phi(x) = y$ has a unique solution x in $B_r(0)$.

Pf: For $x_1, x_2 \in B_r(0)$ ($r > 0$ to be determined)

using remark (1)

consider $\varphi_i(t) = \Psi_i(x_1 + t(x_2 - x_1))$ for $t \in [0,1]$.

Then $\varphi_i(0) = \Psi_i(x_1)$, $\varphi_i(1) = \Psi_i(x_2)$.



$$\varphi_i'(t) = \frac{d}{dt} \Psi_i(x_1 + t(x_2 - x_1))$$
$$= \nabla \Psi_i(x_1 + t(x_2 - x_1)) \cdot (x_2 - x_1)$$

$$\Rightarrow |\Psi_i(x_2) - \Psi_i(x_1)| = |\varphi_i(1) - \varphi_i(0)| = \left| \int_0^1 \varphi_i'(t)\, dt \right|$$
$$\le \int_0^1 |\nabla \Psi_i(x_1 + t(x_2 - x_1)) \cdot (x_2 - x_1)|\, dt$$

$$\leq \left(\int_0^1 |\nabla \Psi_i(x_1+t(x_2-x_1))|\,dt\right)|x_2-x_1|$$

$$\leq |\nabla \Psi_i(x_1+t^*(x_2-x_1))|\,|x_2-x_1|$$

(for some $t^* \in (0,1)$ by Mean Value Thm, since Ψ is C^1)

Note that $x_1, x_2 \in B_r(0) \Rightarrow x_1 + t^*(x_2-x_1) \in B_r(0)$

Let

$$M_r = \sup_{x\in \overline{B_r}(0)} \left(\sum_{i,j=1}^{n} \left|\frac{\partial \Psi_i}{\partial x_j}(x)\right|^2\right)^{1/2} > 0$$

(unless $\Psi \equiv 0$ in $\overline{B_r(0)}$, which is a trivial case.)

$$\Rightarrow \quad |\Psi(x_2)-\Psi(x_1)| = \sqrt{\sum_{i=1}^{n} |\Psi_i(x_2)-\Psi_i(x_1)|^2}$$

$$\leq \sqrt{\sum_{i=1}^{n} |\nabla \Psi_i(x_1+t^*(x_2-x_1))|^2}\;|x_2-x_1|$$

$$\leq M_r\,|x_2-x_1|$$

By $\lim_{x\to 0} \frac{\partial \Psi_i}{\partial x_j}(x) = 0$, $\forall i,j=1,\cdots,n$, and Ψ is C^1,

$\exists\; r>0$ s.t. $M_r \leq \frac{1}{2} \Rightarrow |\Psi(x_2)-\Psi(x_1)| \leq \frac{1}{2}|x_2-x_1|$.

Take $R = (1-\frac{1}{2})r = \frac{r}{2}$. By Thm 3.4 & Remarks (1) & (2),

$\forall\; y \in B_{\frac{r}{2}}(0)$, $\exists\; x \in B_r(0)$ s.t. $\Phi(x) = y$. #

eg 3.7 : Let $g(x) \in C[0,1]$ and $K(x,t) \in C([0,1]\times[0,1])$.

Let $M = \|K\|_\infty = \max_{(x,t)\in[0,1]\times[0,1]} |K(x,t)|$.

Then $\forall\ g \in C[0,1]$ with $\|g\|_\infty < \frac{1}{8M}$,

$\exists$ unique solution $y \in C[0,1]$ with $\|y\|_\infty \leq \frac{1}{4M}$

s.t. $\boxed{y(x) = g(x) + \int_0^1 K(x,t)\,y^2(t)\,dt}$ (Integral Equation)

Pf: Note that $(C[0,1], \|\cdot\|_\infty)$ is a Banach space.

Consider $\Phi = \overline{B_r^\infty(0)} \to C[0,1]$ defined by (r>0 to be determined)

$$y \mapsto \Phi(y) \text{ s.t. } \forall x \in [0,1]$$

$$\Phi(y)(x) = y(x) - \int_0^1 K(x,t)\,y^2(t)\,dt$$

And let $\Psi(y): \overline{B_r^\infty(0)} \to C[0,1]$ be defined by

$$\Psi(y)(x) = -\int_0^1 K(x,t)\,y^2(t)\,dt.$$

Note also $\Phi(0) = 0$ ($\&$ $\Psi(0) = 0$) (where 0 = zero function)

$\forall\ y_1, y_2 \in \overline{B_r^\infty(0)}$,

$$\|\Psi(y_1) - \Psi(y_2)\|_\infty = \max_{x\in[0,1]} \left| -\int_0^1 K(x,t)\,y_1^2(t)\,dt + \int_0^1 K(x,t)\,y_2^2(t)\,dt \right|$$

$$\leq \int_0^1 \left(\max_{x\in[0,1]} |K(x,t)|\right) |y_2^2(t) - y_1^2(t)|\,dt$$

$$\leq M \|y_2^2 - y_1^2\|_\infty$$

$$\leq M \|y_2 + y_1\|_\infty \|y_2 - y_1\|_\infty$$

$$\leq 2rM \|y_2 - y_1\|_\infty$$

Choose $r = \frac{1}{4M}$, then

$$\|\Phi(y_1) - \Phi(y_2)\|_\infty \leq \frac{1}{2} \|y_1 - y_2\|_\infty, \quad \forall\, y_1, y_2 \in \overline{B^\infty_{\frac{1}{4M}}(0)}.$$

Hence Thm 3.4 $\Rightarrow$

$$\forall\, g \in \overline{B^\infty_R(0)} \text{ with } R = (1 - \tfrac{1}{2})r = \tfrac{1}{2} \cdot \tfrac{1}{4M} = \tfrac{1}{8M} > 0,$$

$$\exists!\, y \in \overline{B^\infty_{\frac{1}{4M}}(0)} \text{ s.t. } \Phi(y) = g$$

i.e. $y(x) - \int_0^1 K(x,t)\, y^2(t)\, dt = g(x), \quad \forall\, x \in [0,1]$

which is the required solution to the integral equation. ※

§3.3 The Inverse Function Theorem

Recall: Chain Rule

Let $G: U^{\subset \mathbb{R}^n} \longrightarrow \mathbb{R}^m$, $F: V^{\subset \mathbb{R}^m} \longrightarrow \mathbb{R}^l$ } differentiable,

U, V open in $\mathbb{R}^n$ & $\mathbb{R}^m$ respectively, and

$$G(U) \subset V.$$

Then $H = F \circ G : U \to \mathbb{R}^l$ differentiable and

$$DH(x) = DF(G(x))\, DG(x),$$

where
$$DG(x) = \left(\frac{\partial G_i}{\partial x_j}(x)\right)_{\substack{i=1,\dots,m\\ j=1,\dots,n}} = \begin{pmatrix} -\nabla G_1- \\ \vdots \\ -\nabla G_m- \end{pmatrix} = \begin{pmatrix} \frac{\partial G_1}{\partial x_1} & \cdots & \frac{\partial G_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial G_m}{\partial x_1} & \cdots & \frac{\partial G_m}{\partial x_n} \end{pmatrix}$$

and similarly for DF & DH.

We also need

Prop 3.6 Let $F: B \to \mathbb{R}^n$ be C^1, where B = ball in $\mathbb{R}^n$.

Then $\forall\, x_1, x_2 \in B$,

matrix acts on column vector.

$$F(x_1) - F(x_2) = \left(\int_0^1 DF(x_2 + t(x_1 - x_2))\, dt\right) \cdot (x_1 - x_2)$$

In component form $F = \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix}$, this is

$$F_i(x_1) - F_i(x_2) = \sum_{j=1}^n \left(\int_0^1 \frac{\partial F_i}{\partial x_j}(x_2 + t(x_1 - x_2))\, dt\right)(x_1 - x_2)_j.$$

Pf: For each $i = 1, \cdots, n$,

$$F_i(x_1) - F_i(x_2) = \int_0^1 \left(\frac{d}{dt} F_i(x_2 + t(x_1 - x_2)) \right) dt$$

$$= \int_0^1 \sum_{j=1}^n \left[\frac{\partial F_i}{\partial x_j}(x_2 + t(x_1 - x_2)) \cdot (x_1 - x_2)_j \right] dt$$

$$= \int_0^1 \nabla F_i(x_2 + t(x_1 - x_2)) \bullet (x_1 - x_2)\, dt$$

$$= \left(\int_0^1 \nabla F_i(x_2 + t(x_1 - x_2)) dt \right) \bullet (x_1 - x_2)$$

dot product of vectors

$$\therefore \quad F(x_1) - F(x_2) = \left(\int_0^1 DF(x_2 + t(x_1 - x_2))\, dt \right) \cdot (x_1 - x_2)$$

#

Recall: If $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a point p in an open set U of $\mathbb{R}^n$,

Then $F(p + x) - F(p) = DF(p)x + o(|x|)$

$\forall\, x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ sufficiently small, (i.e. $|x|$ small)

Where $o(|x|)$ is a remaining term such that

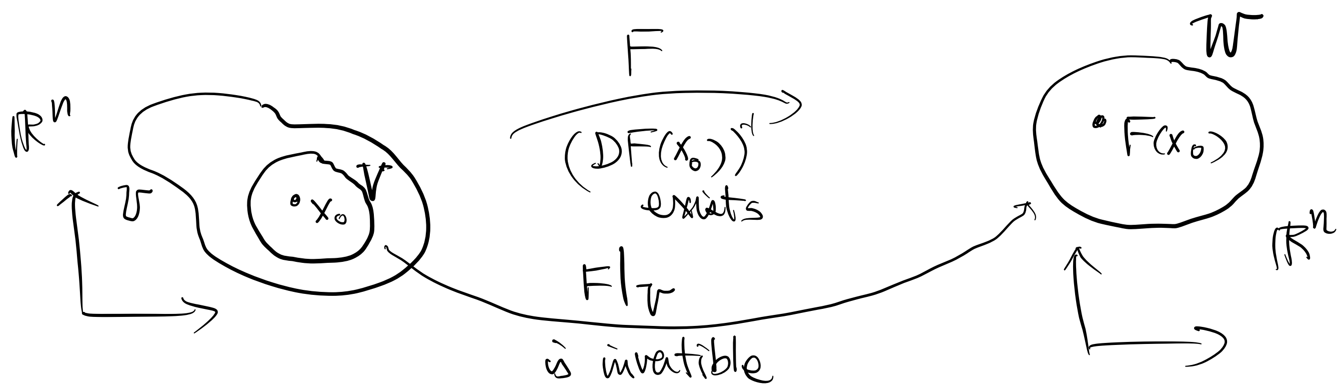
$$\frac{o(|x|)}{|x|} \to 0 \quad \text{as} \quad |x| \to 0.$$

Thm 3.7 (Inverse Function Theorem)

Let $F : U \to \mathbb{R}^n$ be a C^1-map from an open set $U \subset \mathbb{R}^n$. Suppose $x_0 \in U$ and $DF(x_0)$ is invertible (as a matrix or linear transformation).

(a) Then $\exists$ open sets V & W containing x_0 and $F(x_0)$ respectively such that the restriction of F on V is a bijection onto W with a C^1-inverse.

(b) The inverse is C^k when F is C^k, $1 \le k \le \infty$, in V.



Note: We only have local invertibility by the IFT.

Let see some examples before proving the IFT.

eg 3.8: Let $F = (0,\infty) \times (-\infty,\infty) \to \mathbb{R}^2$

$$(r, \theta) \longmapsto (r\cos\theta, r\sin\theta)$$

Then $DF = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$ invertible $\forall (r,\theta)$

Then IFT $\Rightarrow$ F is locally invertible at every point $(r,\theta) \in (0,\infty)\times(-\infty,\infty)$. But F is clearly <u>not</u> <u>globally invertible</u> as it is not one-to-one:

$$F(r,\theta+2\pi) = F(r,\theta).$$

eg 3.9 U = open interval (a,b) in $\mathbb{R}$ $(n=1)$ is a special case:

C^1 function $f:(a,b)\to\mathbb{R}$ with $f'\neq 0$

$\Rightarrow$ f strictly increasing or decreasing

$\Rightarrow$ global inverse exists.

($\therefore$ 1-dim has stronger result than high dimensions)

<u>eg 3.10</u>: (i) $F:\mathbb{R}^2\to\mathbb{R}^2 : (x,y)\mapsto(x^2,y)$.

Then $\quad DF = \begin{pmatrix} 2x & 0 \\ 0 & 1 \end{pmatrix}$ singular at $(x,y)=(0,0)$.

F <u>doesn't</u> satisfy the condition DF invertible in the IFT.
And clearly F is <u>not invertible</u> near $(x,y)=(0,0)$ as

$$F(\pm a,b) = (a^2,b) \quad (\text{2-to-1 near } (0,0)).$$

$\therefore$ "DF invertible" condition can't be removed from IFT.

(ii) $H:\mathbb{R}^n\to\mathbb{R}^n : (x,y)\mapsto(x^3,y)$ is bijective

& $H^{-1}(x,y) = (x^{1/3},y)$ exists.

But $DH = \begin{pmatrix} 3x^2 & 0 \\ 0 & 1 \end{pmatrix}$ singular at $(x,y)=(0,0)$.

The point is: H^{-1} is not C^1 near $(x,y)=(0,0)$.

$\therefore$ "DF invertible" is only a "sufficient" condition for "local invertibility".

Terminology: The condition in IFT that $DF(x_0)$ is invertible is called the nondegeneracy condition.

By eg 3.10, without nondegeneracy condition, the map may or may not be local invertible.

But Nondegeneracy condition is necessary for the differentiability of the local inverse:

Prop 3.8: Let $F: U \overset{\subset \mathbb{R}^n \text{ open}}{} \longrightarrow \mathbb{R}^n$ be C^1, and $x_0 \in U$.

Suppose $\exists$ open V s.t. $x_0 \in V \subset U$, and $F|_V$ has a differentiable inverse.

Then $DF(x_0)$ is non-singular. (ie. invertible).

Pf: Suppose the local inverse $(F|_V)^{-1}$ exist and is differentible at the point $y_0 = F(x_0)$.

Then Chain rule $\Rightarrow$ $D(F^{-1})(y_0)\, DF(x_0) = \text{Identity}$

$\Rightarrow$ $DF(x_0)$ is invertible. #

Proof of IFT (Thm 3.7)

Special Case: $x_0 = 0$, $y_0 = F(x_0) = F(0) = 0$,

$DF(0) = I$ (the Identity)

Step 1 Let $\Psi(x) = -x + F(x)$.

Then $\exists r > 0$ s.t.

$$|\Psi(x_2) - \Psi(x_1)| \leq \frac{1}{2} |x_2 - x_1| \text{ on } \overline{B_r(0)}.$$

Pf of Step 1: As $0 \in U$ and U is open, $\exists r_0 > 0$ s.t. $\overline{B_{r_0}(0)} \subset U$. Then

$$\Psi(x_1) - \Psi(x_2) = -x_1 + F(x_1) + x_2 - F(x_2)$$

(prop. 3.6)
$$= \left(\int_0^1 DF(x_2 + t(x_1 - x_2))\,dt\right)(x_1 - x_2) - (x_1 - x_2)$$

$$= \left[\int_0^1 DF(x_2 + t(x_1 - x_2))\,dt - I\right](x_1 - x_2).$$

$$= \int_0^1 [DF(x_2 + t(x_1 - x_2)) - DF(0)]\,dt\,(x_1 - x_2)$$

As F is C^1, we have $\forall \varepsilon > 0$, $\exists r > 0$, $(r \leq r_0)$ such that

$$\|DF(x) - DF(0)\| < \varepsilon, \quad \forall x \in \overline{B_r(0)},$$

where $\|(b_{ij})\| = \sqrt{\sum_{i,j} b_{ij}^2}$ for any $n \times n$ matrix (b_{ij}).

Since $\overline{B_r(0)}$ is convex,

$$x_1, x_2 \in \overline{B_r(0)} \Rightarrow x_2 + t(x_1 - x_2) \in \overline{B_r(0)}.$$

Hence $\forall \varepsilon > 0$, $\exists r > 0$ $(r \leq r_0)$ such that

$$\| DF(x_2 + t(x_1 - x_2)) - DF(0) \| < \varepsilon, \quad \forall x_1, x_2 \in \overline{B_r(0)} \text{ and } t \in (0,1).$$

Therefore
$$|\Psi(x_1) - \Psi(x_2)| \leq \varepsilon |x_1 - x_2|.$$

Choosing $\varepsilon = \frac{1}{2} > 0$, then $\exists r > 0, (r \leq r_0)$ s.t.

$$|\Psi(x_1) - \Psi(x_2)| \leq \frac{1}{2} |x_1 - x_2|, \quad \forall x_1, x_2 \in \overline{B_r(0)}.$$

This completes the proof of Step 1 of the special case. #

Step 2 $r > 0$ as in Step 1. Then

$$\forall y \in B_{\frac{r}{2}}(0), \quad \exists x \in B_r(0) \text{ such that } F(x) = y.$$

And the local inverse G of F,

$$G : B_{\frac{r}{2}}(0) \to G(B_{\frac{r}{2}}(0)) \subset B_r(0)$$

satisfies

$$|G(y_1) - G(y_2)| \leq 2|y_1 - y_2|, \quad \forall y_1, y_2 \in B_{\frac{r}{2}}(0).$$

with $G(B_{\frac{r}{2}}(0))$ open in $B_r(0)$.

Pf of Step 2: By Step 1, one can apply Thm 3.4 (Perturbation of Identity) to show that $\forall y \in \overline{B_R(0)}$ with $R = (1 - \frac{1}{2}) \cdot r = \frac{r}{2}$

$\exists\ x \in \overline{B_r(0)}$ s.t. $F(x) = y$.

Then remark (2) (after the proof of Thm3.4)

$\Rightarrow \forall y \in B_R(0),\ \exists\ x \in B_r(0)$ s.t. $F(x) = y$.

and by remark (3) (after the proof of Thm3.4) (which is the Remark 3.1 of Prof Chou's notes).

we have

$$|G(y_1) - G(y_2)| \le \frac{1}{1-\frac{1}{2}} |y_1 - y_2|$$

$$= 2|y_1 - y_2| \quad \forall y_1, y_2 \in \overline{B_{\frac{r}{2}}(0)}.$$

Finally, remark (2) again $\Rightarrow$ $G(B_{\frac{r}{2}}(0))$ is open in $B_r(0)$. #

Step3 G is differentiable on $B_{\frac{r}{2}}(0)$ and

$DG(y) = (DF)^{-1}(G(y)),\ \forall y \in B_{\frac{r}{2}}(0)$.

Pf of Step3: As $DF(0) = I$, we may assume that $DF(x)$ is invertible $\forall x \in B_r(0)$ for the $r > 0$ given in step 1 (Since we may always choose a smaller r in the proof of Step 1.)

Let $W = B_{\frac{r}{2}}(0) \left(= B_R(0) \right)$ and

$V = G(B_{\frac{r}{2}}(0)) = G(W) \ni 0$.

Then $\quad G: W \to V \qquad$ (and $F: V \to W$)

(By Chain rule, if G is differentiable, then

$$DF(G(y))\,DG(y) = I, \quad \forall y \in W$$

hence $\quad DG(y) = (DF)^{-1}(G(y))$.

So we target $(DF)^{-1}(G(y))$ as the required linear map in differentiability)

For $y_1 \in W = B_{\frac{r}{2}}(0)$ & $y_1 + y \in W = B_{\frac{r}{2}}(0)$,

we have $\quad y = (y_1 + y) - y_1 = F(G(y_1+y)) - F(G(y_1))$

Denote $\quad x_1 = G(y_1+y)$ and $x_2 = G(y_1)$

Then
$$\begin{aligned} y &= F(x_1) - F(x_2) \\ &= \left[\int_0^1 DF(x_2 + t(x_1-x_2))\,dt\right](x_1-x_2) \qquad \text{(Prop 3.6)} \\ &= \int_0^1 \left[DF(x_2+t(x_1-x_2)) - DF(x_2)\right]dt\,(x_1-x_2) + DF(x_2)(x_1-x_2) \end{aligned}$$

Hence

$$(x_1-x_2) = (DF)^{-1}(x_2)\,y - (DF)^{-1}(x_2)\int_0^1 \left[DF(x_2+t(x_1-x_2)) - DF(x_2)\right]dt\,(x_1-x_2)$$

i.e.

$$G(y_1+y) - G(y_1) = (DF)^{-1}(G(y_1))\,y + R, \qquad \left(\begin{aligned} x_1 &= G(y_1+y) \\ x_2 &= G(y_1)\end{aligned}\right)$$

where $\quad R = (DF)^{-1}(x_2)\int_0^1 \left[DF(x_2) - DF(x_2+t(x_1-x_2))\right]dt\,(x_1-x_2)$

Observes that

$$|x_1 - x_2| = |G(y_1+y) - G(y_1)| \leq 2\,|(y_1+y) - y_1| = 2|y|, \quad (\text{step}2)$$

we have, $|x_1 - x_2| \to 0$ as $|y| \to 0$ and

$$\frac{|R|}{|y|} \leq 2\,\|DF^{-1}(x_2)\| \int_0^1 \underbrace{\|DF(x_2) - DF(x_2 + t(x_1 - x_2))\|}_{\to 0 \text{ as } |x_1 - x_2| \to 0}\,dt \quad \left(F \in C^1\right)$$

By assumption F is C^1 ($x_1, x_2 \in \overline{B_r(0)}$), we have

$$\lim_{|y|\to 0} \frac{|R|}{|y|} = 0 .$$

Therefore $G(y_1+y) - G(y) = (DF)^{-1}(G(y_1))\,y + o(|y|)$,

which implies G is differentiable at $y_1 \in B_{\frac{r}{2}}(0) = W$

and $DG(y_1) = (DF)^{-1}(G(y_1))$. #