3.5 Appendix: Completion of a Metric Space

Def: A metric space (X, d) is said to be <u>isometrically embedded</u> in metric space (Y, ρ) if

$\exists$ a mapping $\Phi: X \to Y$ s.t.

$$d(x,y) = \rho(\Phi(x), \Phi(y)).$$

<u>Notes</u>: (i) Φ is called an <u>isometric embedding</u> from (X, d) into (Y, ρ). And sometime called a <u>metric preserving map</u>.

(ii) Φ must be one-to-one and continuous.

Def: Let (X, d) and (Y, ρ) be metric spaces. We call (Y, ρ) a <u>completion</u> of (X, d) if (1) (Y, ρ) is complete.

(2) $\exists$ isometric embedding

$$\Phi: (X, d) \to (Y, \rho)$$

such that the closure $\overline{\Phi(X)} = Y$.

eg: $(Y,\rho) = (\mathbb{R}, \text{standard})$, $X = \mathbb{Q} \subset \mathbb{R}$

$(X,d) = (\mathbb{Q}, \text{induced metric})$

Then
- $(\mathbb{R}, \text{standard})$ is complete;
- $\Phi : (\mathbb{Q}, \text{induced metric}) \to (\mathbb{R}, \text{standard})$

$$\begin{array}{ccc} \mathbb{Q} & & \mathbb{R} \\ \cup & & \cup \\ q & \longmapsto & q \end{array}$$

- $\overline{\Phi(\mathbb{Q})} = \overline{\mathbb{Q}} = \mathbb{R}$ ($\mathbb{Q}$ is dense in $\mathbb{R}$)

Def: Two metric spaces (X,d), (X',d') are called *isometric* if $\exists$ *bijective* isometric embedding from (X,d) onto (X',d').

Notes: (i) the inverse of the bijective isometric embedding is also an isometric embedding.

(ii) Two metric spaces will be regarded as the *same* if they are isometric.

Thm: If (Y,ρ) & (Y',ρ') are completions of a metric space (X,d). Then (Y,ρ) and (Y',ρ') are isometric.

i.e. Completion is unique up to isometry.

§3.2 The Contraction Mapping Principle

Def: (1) Let (X, d) be a metric space. A map $T:(X,d) \to (X,d)$ is called a <u>contraction</u> if $\exists$ constant $\gamma \in (0,1)$, such that
$$d(Tx, Ty) \le \gamma\, d(x,y), \quad \forall x, y \in X.$$

(2) A point $x \in X$ is called a <u>fixed point</u> of T if $Tx = x$. (Usually write Tx instead of $T(x)$)

<u>Thm 3.3</u> (<u>Contraction Mapping Principle</u>)
Every <u>contraction</u> in a <u>complete</u> metric space admit a <u>unique</u> <u>fixed point</u>.
(This is also called the <u>Banach Fixed Point Thm</u>)

<u>Pf</u>: <u>Uniqueness</u>: Suppose x & y are fixed pts. of T.
Then
$$d(x,y) = d(Tx, Ty) \quad (x, y \text{ are fixed by } T)$$
$$\le \gamma\, d(x,y) \text{ for some } \gamma \in (0,1).$$
$$(T \text{ contraction})$$
$$\Rightarrow d(x,y) = 0 \Rightarrow x = y.$$

Existence: Let $x_0 \in \mathbb{X}$.

Define $\{x_n\}_{n=1}^{\infty}$ by $\quad x_n = Tx_{n-1}$, for $n=1,2,\ldots$

Then $\quad x_n = Tx_{n-1} = T(Tx_{n-2}) = T^2 x_{n-2}$

$$= \cdots = T^n x_0 .$$

For any $n \geq N$,

$$d(x_n, x_N) = d(T^n x_0, T^N x_0) = d\left(T^{(n-N)+N} x_0, T^N x_0\right)$$

$$= d\left(T(T^{(n-N)+N-1} x_0), T(T^{N-1} x_0)\right)$$

$$\leq \gamma\, d(T^{(n-N)+N-1} x_0, T^{N-1} x_0)$$

(where $\gamma \in (0,1)$ is the constant s.t. $d(Tx, Ty) \leq \gamma d(x,y)$, $\forall x, y \in \mathbb{X}$)

$$\leq \cdots$$

$$\leq \gamma^N d(T^{(n-N)} x_0, x_0)$$

$$\leq \gamma^N \Big[d(T^{(n-N)} x_0, T^{(n-N)-1} x_0) + d(T^{(n-N)-1} x_0, T^{(n-N)-2} x_0)$$
$$+ \cdots + d(Tx_0, x_0) \Big]$$

$$\leq \gamma^N \Big[d(Tx_0, x_0) + \gamma d(Tx_0, x_0) + \cdots$$
$$+ \gamma^{(n-N)-2} d(Tx_0, x_0) + \gamma^{(n-N)-1} d(Tx_0, x_0) \Big]$$

$$= \gamma^N \left[1 + \gamma + \cdots + \gamma^{(n-N)-1} \right] d(Tx_0, x_0)$$

$$< \frac{\gamma^N}{1-\gamma} d(Tx_0, x_0)$$

Therefore, $\forall \varepsilon > 0$, if $N > 0$ is chosen s.t.

$$\frac{\gamma^N}{1-\gamma} d(Tx_0, x_0) < \frac{\varepsilon}{2},$$

we have $\forall\, n, m \geq N$,

$$d(x_n, x_m) \leq d(x_n, x_N) + d(x_N, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\therefore$ $\{x_n\}$ is a Cauchy seq. in $(\mathbb{X}, d)$.

By completeness of $(\mathbb{X}, d)$, $\exists\, x \in \mathbb{X}$ s.t. $x_n \to x$.

Note that a contraction is always continuous (Ex!) we have

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_{n-1} = T \lim_{n\to\infty} x_{n-1} = Tx.$$

$\therefore$ x is a fixed point of T. ※

eg 3.3 $\quad T: (0,1] \to (0,1]$ $\quad$ (Caution: $(0,1]$ is not complete)

$$x \mapsto \frac{x}{2}.$$

Clearly $|Tx - Ty| = \frac{1}{2}|x-y|$ $\quad (\gamma = \frac{1}{2} < 1)$

$\therefore$ T is a contraction.

However, if $x \in (0,1]$ is a fixed point of T,

then $\quad Tx = x \iff \frac{x}{2} = x \iff x = 0 \notin (0,1]$.

$\therefore$ T has no fixed point on $(0,1]$.

This example shows that "completeness" is necessary in the Contraction Mapping Principle.

eg 3.4 : $S: \mathbb{R} \to \mathbb{R}$ ($\mathbb{R}$ is complete)

$x \mapsto x - \log(1+e^x)$.

Then $\frac{dS}{dx} = 1 - \frac{e^x}{1+e^x} = \frac{1}{1+e^x} > 0$

$\Rightarrow |Sx - Sy| = \left|\frac{dS}{dx}(c)\right| |x-y| < |x-y|$

(But there is no constant $\gamma < 1$ such that $|Sx - Sy| \leq \gamma |x-y|$ (Ex!))

Since $-\log(1+e^x) \neq 0 \ \forall x \in \mathbb{R}$,

$Sx \neq x \ \forall x \in \mathbb{R}$ i.e. no fixed point

This example shows that $\gamma < 1$ cannot replaced by $\gamma \leq 1$ #

eg 3.5 Let $f: [0,1] \to [0,1]$ continuously differentiable with $|f'(x)| < 1$ on $[0,1]$. Then f has a fixed point in $[0,1]$.

Pf : By mean value theorem

$\forall x, y \in [0,1]$, $\exists z \in [0,1]$ s.t.

$$f(x) - f(y) = f'(z)(x-y)$$

$$\Rightarrow |f(x) - f(y)| \leq |f'(z)| |x-y|$$

$$\leq \left(\sup_{[0,1]} |f'(z)|\right) |x-y|.$$

Since $|f'(z)| < 1$ & $f'(z)$ cts on $[0,1]$,

$$\gamma = \sup_{[0,1]} |f'(z)| \in [0,1).$$

If $\gamma = 0$, then $f \equiv c$ on $[0,1] \Rightarrow f(c) = c$.

If $\gamma \neq 0$, then $\gamma \in (0,1)$ & $|f(x)-f(y)| \leq \gamma |x-y|$ $\forall x,y \in [0,1]$.

$\Rightarrow$ f is a contraction on the complete metric space $([0,1], \text{standard})$.

By contraction mapping principle, f has a fixed point. #