§2.5 Elementary Inequalities for Functions

Recall

> **Young's Inequality**
>
> For $a, b > 0$ and $p > 1$,
>
> $$ab \le \frac{a^p}{p} + \frac{b^q}{q}, \qquad \text{where } q \text{ is given by } \frac{1}{p} + \frac{1}{q} = 1$$
>
> and "equality holds" $\iff a^p = b^q$.

Note: $q = \frac{p}{p-1} > 1$ is called the conjugate of p.

(Recall Pf: Study the minimum of

$$\varphi(a) = \frac{a^p}{p} + \frac{b^q}{q} - ab. \quad (\text{Ex!})$$)

Note: If $p = 2$, it is the elementary inequality

$$2ab \le a^2 + b^2.$$

Thm 2.10 (Hölder's Inequality)

Let $f, g \in R[a,b]$ (Riemann integrable) and $p>1$.

Then
$$\int_a^b |f(x)g(x)|\,dx \le \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q dx\right)^{\frac{1}{q}},$$

where $q = \frac{p}{p-1}$ the conjugate of p.

"Equality holds"

$\Leftrightarrow$ either (a) f or $g = 0$ almost everywhere,

or (b) $\exists$ constant $\lambda > 0$ s.t.

$|g(x)|^q = \lambda |f(x)|^p$ almost everywhere.

($\Leftrightarrow$ $\exists$ constants $\lambda_1, \lambda_2 \ge 0$, not both zero, such that
$\lambda_1 |f(x)|^p = \lambda_2 |g(x)|^q$ a.e.)

Pf: Omitted.

Note: If we denote $\|f\|_p = \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}}$. Then the Hölder Inequality can be written as

$$\int_a^b |f(x)g(x)|\,dx \le \|f\|_p \|g\|_q .$$

Note: Limiting cases

(Note: Riemann integrable $\Rightarrow$ bounded)

(i) $p \to 1$ ($\Rightarrow q \to +\infty$)

$$\int_a^b |f(x)g(x)|\,dx \le \|f\|_1 \, \|g\|_\infty$$

(ii) $p \to +\infty$ ($\Rightarrow q \to 1$)

$$\int_a^b |f(x)g(x)|\,dx \le \|f\|_\infty \|g\|_1$$

Thm 2.11 (Minkowski's Inequality)

$\forall\, f, g \in R[a,b]$, and $p > 1$,

$$\|f+g\|_p \le \|f\|_p + \|g\|_p .$$

"Equality holds"

$\Leftrightarrow$ either (a) f or $g = 0$ a.e.

or (b) $\|f\|_p > 0$, $\|g\|_p > 0$ and $\exists$ constant $\lambda > 0$ s.t. $g(x) = \lambda f(x)$ a.e.

($\Leftrightarrow$ $\exists$ constants $\lambda_1, \lambda_2 \ge 0$, not both zero, s.t. $\lambda_1 f(x) = \lambda_2 g(x)$ a.e.)

Pf: Omitted.

Remark: This shows that $\|f\|_p$ for $p>1$ is a norm

on $R[a,b]/\sim$ ($\leftarrow$ relation mod a.e.)

(other conditions are trivial)

Ch3 The Contraction Mapping Principle

§3.1 Complete Metric Space

Def: Let (X, d) be a metric space.

(1) A sequence $\{x_n\}$ in (X, d) is a Cauchy sequence if $\forall \varepsilon > 0$, $\exists n_0$ s.t. $d(x_n, x_m) < \varepsilon$, $\forall n, m \geq n_0$.

(2) (X, d) is complete if every Cauchy sequence in (X, d) converges.

(3) A subset E is complete if the induce metric subspace (E, d) is complete. (i.e. $d = d|_{E \times E}$)
i.e. every Cauchy sequence in E converges with limit in E.

Note: Convergent sequence is a Cauchy sequence (Ex!)

Prop 3.1 Let (X, d) be a metric space.

(a) Every complete set in X is closed.

(b) If X is complete, then every closed set in X is complete.

Pf: (a) Let $E \subset X$, & E complete.

Suppose $\{x_n\} \subset E$ with $x_n \to x$ in X.

By note, $\{x_n\}$ is a Cauchy seq. in E

Then completeness of $E \Rightarrow x_n \to z \in E$.

Uniqueness of limit $\Rightarrow x = z \in E$

$\therefore$ E is closed.

(b) Let (X, d) be complete & E is closed in X.

Then every Cauchy seq. $\{x_n\}$ in E is a Cauchy seq in X. Completeness of $X \Rightarrow \exists x \in X$, s.t. $x_n \to x$. Since E is closed, $x \in E$.

$\therefore$ E is complete. ※

eg 3.1:

- $(\mathbb{R}$, standard$)$ is complete
- $[a,b]$, $(-\infty, b]$, $[a, \infty)$ complete
- $[a, b)$ (b finite) not complete
  $\left(\because x_n = b - \frac{1}{n} \to b \notin [a,b)\right)$
- $\mathbb{Q}$ is not complete.

eg 3.2 $(X = C[a,b], d_\infty)$ is complete:

Cauchy seq $\{f_n\}$ in d_∞-metric

$\Leftrightarrow \forall \varepsilon > 0, \exists n_0$ s.t.

$$\max_{[a,b]} |f_n(x) - f_m(x)| < \varepsilon, \quad \forall n, m \geq n_0$$

$\therefore$ $f_n(x) \rightarrow f(x)$ uniformly for some $f \in C[a,b]$

$\#$

eg Let $P = \{ f \in C[a,b] : f(x) = p(x)$ on $[a,b]$ for some polynomial $p(x) \}$.

Then P is not complete (in d_∞-metric):

$$h_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!} \in P$$

but $h_n(x) \rightarrow e^x$ uniformly (in d_∞-metric)

& $e^x \notin P$.