MATH3060 HW2 Due date: Oct 8, 2021 (at 12:00 noon)

---

1. Let f be a function defined on $(-a,a)$ for some $a>0$.
   (a) Show that f is Lipschitz continuous at $x=0$ if both its left and right derivatives exist at $x=0$.
   (b) Show that (a) is only a sufficient condition but not necessary by constructing a function which is Lipschitz continuous at $x=0$ and its one sided derivatives do not exist.

2. Let f be a Riemann integrable (2π-periodic) function on $[-\pi,\pi]$ with Fourier coefficients a_n & b_n. Show that, by assuming uniform convergence,

$$a_0+\sum_{k=1}^{\infty} r^n(a_n\cos nx+b_n\sin nx)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-r^2}{1-2r\cos y+r^2}f(x+y)\,dy$$

for any $0\leq r<1$.

3. Using Weierstrass Approximation Theorem to show that there is a countable subset of $C[a,b]$ the space of continuous functions on the interval $[a,b]$ such that for any $f\in C[a,b]$ and $\varepsilon>0$, there exists g in the subset such that

$$\|f-g\|_\infty<\varepsilon.$$

4. Show that

$$x^3 - \pi^2 x \sim \sum_{n=1}^{\infty} (-1)^n \frac{12}{n^3} \sin nx \qquad (x \in [-\pi, \pi])$$

and by Parseval's Identity that

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$$

(End)