Ch 1 Fourier Series

Def: (1) Trigonometric Series (三角級數) on $[-\pi, \pi]$ is a series of functions of the form

$$\sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (\text{where } a_n, b_n \in \mathbb{R})$$
$$= a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (b_0 = 0)$$

(2) If $b_n = 0$, $\forall n$, it is called a cosine series

If $a_n = 0$, $\forall n$, it is called a sine series

Easy facts

(1) If $\sum_{n=0}^{\infty} |a_n|$, $\sum_{n=0}^{\infty} |b_n| < \infty$

then $\sum_{n=0}^{\infty} a_n \cos nx + b_n \sin nx$

is uniformly and absolutely convergent

In particular, if $|a_n|, |b_n| \le \frac{C}{n^s}$, $s > 1$ (for some $C > 0$)

then $\sum_{n=0}^{\infty} |a_n|$, $\sum_{n=0}^{\infty} |b_n| < \infty$ and hence

$\sum_{n=0}^{\infty} a_n \cos nx + b_n \sin nx$ is uniformly and absolutely convergent

(Pf: By M-test & $|\cos nx|, |\sin nx| \le 1$)

(2) In this case,

$\phi(x) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} a_n \cos nx + b_n \sin nx$ is continuous on $[-\pi, \pi]$.

(3) $\phi(x)$ defined in (2) is 2π-periodic

Pf: $\phi(x+2\pi) = \lim_{n\to\infty} \sum_{k=0}^{n} \left[a_k \cos(k(x+2\pi)) + b_k \sin(k(x+2\pi))\right]$

$= \lim_{n\to\infty} \sum_{k=0}^{n} a_k \cos kx + b_k \sin kx$

$= \phi(x)$ #

Def: Let f be a 2π-periodic function on $\mathbb{R}$ which is Riemann integrable on $[-\pi,\pi]$. Then the Fourier Series (or Fourier expansion) of f is the trigonometric series

$$a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx)$$

with
$$\begin{cases} a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\,dy \\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\cos ny\,dy \\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\sin ny\,dy \end{cases}$$
$(n \geq 1)$

Fourier Coefficients of f

Notes

(1) a_0 = average of f over $[-\pi,\pi]$

(2) Fourier series depends on the global information of f on $[-\pi,\pi]$.

(3) $f_1 \equiv f_2$ "almost everywhere" on $[-\pi,\pi]$

$\Rightarrow$ f_1 & f_2 have the same Fourier Series.

(4) Fourier series of f depends only on $f|_{(-\pi,\pi)}$, independent of the values of f on the end points.

($f_1 \equiv f_2$ "almost everywhere" means meas$(\{f_1 \neq f_2\}) = 0$,
i.e. $\forall \varepsilon > 0$, $\exists$ open intervals I_n, $n=1,2,\cdots$ s.t. $\{f_1 \neq f_2\} \subset \bigcup_{n=1}^{\infty} I_n$, $\sum_{n=1}^{\infty} |I_n| < \varepsilon$)