MATH1510E (Wk1.1,1.2)

Keywords: Transcendental functions, other examples of functions (Monday)

Domain, range, function, examples; (Wednesday)

Properties of functions: one-one, onto (Wednesday)

Sequences & functions, examples (Wednesday)

Some trigo. Identities (Wednesday), else.

Transcendental function – this is the first concept in the syllabus, so we discuss this "name" here.

First we have (1) polynomial functions, they are objects written in the form $a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + a_n x^n$ (This kind of expressions is called "degree *n* polynomial", provided $a_n \neq 0$).

Rational functions

These are functions of the form $\frac{polynomial}{polynomial}$.

Example

$$\frac{1+3x+x^4}{2-3x+x^3}$$

Transcendental Functions

Transcendental functions include: (i) trigonometric functions like $\sin(x)$, $\cos(x)$, $\tan(x)$, or $\sec(x)$, $\csc(x)$, $\cot(x)$;
(ii) exponential function, logarithm function & (iii) hyperbolic functions (see below for explanation).

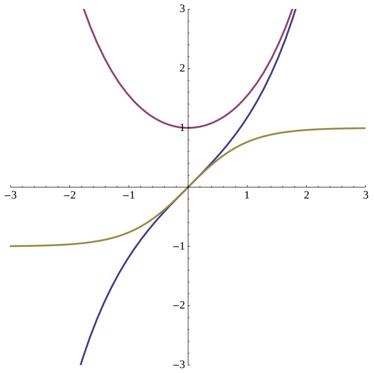
Properties These functions <u>cannot be written</u> as polynomials!
The name "transcendental functions" include also functions like (ii) $\ln(x)$, e^x as well as the hyperbolic functions defined by the formulas (iii) $\sinh(x) = \frac{e^x - e^{-x}}{2}$ and $\cosh(x) = \frac{e^x + e^{-x}}{2}$

Remark
Make sure that you know the picture of each of the trigo. functions.

Pictures of the Hyperbolic Functions

$$\sinh(x), \cosh(x), \tanh(x)$$

Question: In the following picture, can you recognize which is which?



Function, Domain, Range

A function, say f, is a rule giving a unique value, $f(x)$, to a given value x.

The collection of all such x is called Domain of f. (Notation: Dom(f))
The collection of all such $f(x)$ is called the Range of f. (Notation: R(f) or Range(f)).
The word "codomain" is about any set containing ("just" or "more than") all those elements in the range of a function.

Remark

- The rule for a function is usually written using "a single letter" or "several letters" & without "(x)" or "(t)". E.g. g, sin , exp.
- When we put "(x)" (or "(t)") after the symbol, e.g. after g, sin , exp , we call it the "value of g at x", or "value of sin at x", etc.

Examples/Short Questions

1. Let f be a function assigning (i.e. "giving") to each month of the year the initial alphabet of that month.
   What is $\mathrm{Dom}(f)$? What is $\mathrm{Range}(f)$?
2. Let f be a function from the domain $\mathbb{R}$ to the target (or "codomain") $\mathbb{R}$ defined by the rule: $f(x) = \frac{x}{x^2+1}$. Find

   (i) $f(-1)$,

   (ii) $f\Big(f\big(f(-1)\big)\Big)$.

Abstract Picture(s) for function(s)

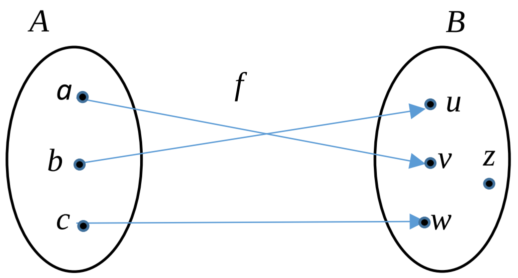
In school math, functions are usually given by one-line formulas. But actually a function can be defined by very complicated formulas, e.g.

$$\text{abs(x)} = \begin{cases} x, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -x, & \text{if } x < 0 \end{cases}$$

is a function defined by a three-line formula. (Notation: Traditionally, we write $|x|$ for this function.)

Now the abstract picture for a function.

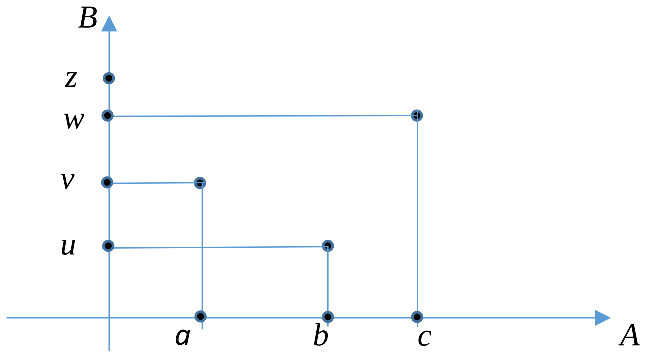
We usually visualize a function by the following kind of diagram:



Here the domain is A, the codomain (or "target") is B and $f(a) = v, f(b) = u, f(c) = w$. Note also that each element in A is being sent to some element in B, but ***not every*** element in B has "pre-image point(s)".

If each element in the codomain has preimage point(s), then we say the function is "onto".

School Math Way of Visualizing it



Inverse Function

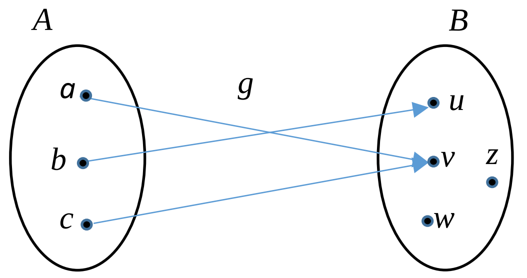
Let $f: A \to B$ be a function. Suppose also that f is onto, (i.e. each element, say "y", comes from some pre-image point(s) x, i.e. $f(x) = y$, every time when a "y" is taken from the codomain B), and if we have one more condition (see below for this), then we can go backward and define the "inverse function", which has the notation f^{-1}, of the function f. The "inverse" function has the rule:

$$f^{-1}(y) = x.$$

The extra condition needed is:

One-one function

A function is a one-one function, if whenever $f(x_1) = f(x_2)$, then the two points x_1 and x_2 must be the same point (i.e. $x_1 = x_2$). A common sense way of describing a one-one function is "every point $f(x)$ in the range has one and only one preimage point").



Example

The function f on p.3 is a one-one function, but the function g in the above picture is not one-one. Why?

Reason We look at the range of g. $Range(g) = \{u, v\}$. Now among these two elements u & v, $u = g(b)$ (i.e. it has one pre-image point), but $v = g(a) = g(c)$ so both the point a and c are assigned the same value v.

Examples

In the following, $\mathbb{N} = \{0,1,2,\cdots\}$, the set of natural numbers starting from zero.

1. Find a function $f\colon \mathbb{N} \to \mathbb{R}$ satisfying (*) f is onto but not one-one.
2. Find a function $f\colon \mathbb{N} \to \mathbb{R}$ satisfying (*) f is one-one but not onto.
3. Find the range of the function $f\colon \mathbb{R}\backslash\{9\} \to \mathbb{R}$ given by $f(x) = \frac{x^2+9}{x-9}$.

Hint for 3. The main idea is: "find (for which) y" is the equation

$y = \frac{x^2+9}{x-9}$ "solvable".

The following trigonometric identities (and many more) will be useful in the course.

Some Useful Trigonometric Identities

(1) $\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x$

(2) $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$etc.$

By letting $x + y = A, x - y = B$, one obtains from the first two formulas the following:

$$\sin(A) + \sin(B) = 2 \sin x \cos y$$
$$= 2 \sin(\ (A+B)/2\ )\cos\ ((A-B)/2)$$

Similarly, one obtains formulas for $\sin(A) - \sin(B),\ \cos(A) \pm \cos(B)$

Quick Proof of (1) & (2)

One can get a diagram-free quick proof of these identities using Euler's Formula (*), i.e. $$e^{ix} = \cos x + i \sin x$$
where x is measured in "radian" and $i = \sqrt{-1}$.

Applying (*) twice, we get

$$e^{iA} = \cos A + i \sin A$$

and

$$e^{iB} = \cos B + i \sin B$$

Multiplying them together, we obtain

$$e^{i(A+B)} = e^{iA}e^{iB} = (\cos A + i \sin A)(\cos B + i \sin B)$$
$$= \cos A \cos B + i \sin A \cos B + i \sin B \cos A + \underbrace{i\,i}_{-1} \sin A \sin B$$
$$= \cos A \cos B - \sin A \sin B + i\,(\sin A \cos B + \sin B \cos A)$$

But remember that (Euler's formula again!)

$$e^{i(A+B)} = \cos(A+B) + i \sin(A+B)$$

So we obtain

$$\cos(A+B) + i \sin(A+B) = \cos A \cos B - \sin A \sin B + i\,(\sin A \cos B + \sin B \cos A)$$

Comparing the terms (with and without i attached) on the left-hand & right-hand sides of the "equal" sign, we obtain

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

and $$\sin(A+B) = \sin A \cos B + \sin B \cos A$$

After defining function and mentioning some of its properties, let's mention a very special type of function, which you've already learned in school. It is "sequence".

Sequence

A sequence is an <u>ordered</u> list of objects (= <u>numbers</u> in this course).

Each of these objects is traditionally denoted by x_n. The subscript means the n^{th} object.

Example

Consider the sequence given by $x_1 = 1$ and $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$. If we know that as $n \to \infty$ ("$n \to \infty$" = abbreviation for the phrase "n goes to positive infinity"), the numbers x_n goes to some limiting number, say $L,$ then we can argue as follows:

$$L = \frac{1}{2}\left(L + \frac{2}{L}\right)$$

Solving this equation for L, we obtain $L = \sqrt{2}$.

Another Way to think about Sequence

One can also think of a sequence as a special kind of function, namely a function whose domain is the set of natural numbers, denoted by the symbol $\mathbb{N}$ (or a subset of $\mathbb{N}$, for example, the set $\{1,2,3,\cdots\}, \{2,3,4,...\}$ or $\{k, k+1, k+2, \cdots\}$.)

Remark

In this course, we assume that the symbol $\mathbb{N}$ means the set $\{0,1,2,3,\cdots\}$.

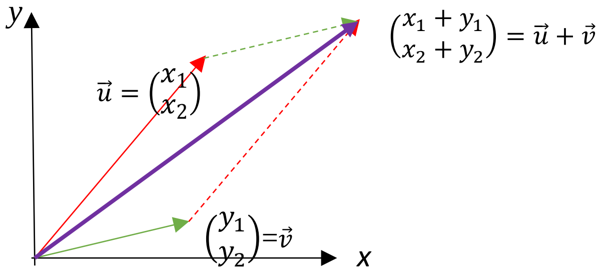
Vectors in 2D, 3D.

A vector in 2D is an ordered pair of numbers written in the form $\binom{x_1}{x_2}$, where x_i denotes the i^{th} component of the vector.

Adding, Subtracting, Scalar multiplying Vectors, Norm

Addition/Subtraction of Two vectors

Suppose $\binom{x_1}{x_2}, \binom{y_1}{y_2}$ are 2 vectors in $\mathbb{R}^2$, their sum/difference is then the vector given by $\binom{x_1}{x_2} \pm \binom{y_1}{y_2} = \binom{x_1 \pm y_1}{x_2 \pm y_2}$



Remark

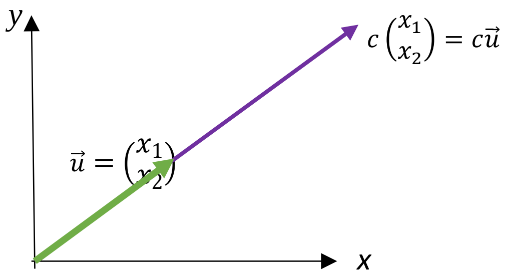
Similar formula holds for sum/difference of two vectors in $\mathbb{R}^3$.

Scalar Multiplication of a Vector by a Scalar

Scalar mult. = geometrically "scaling up/down" a vector.

Given a vector $\binom{x_1}{x_2}$ in $\mathbb{R}^2$ and a scalar, say c, the scalar multiplication of the vector with the scalar c is the "new" vector $c\binom{x_1}{x_2} = \binom{cx_1}{cx_2}$.

That means, we "scale up" (= make "longer") or "scale down" (= make "shorter") each component of the vector by the same factor c.



Remark

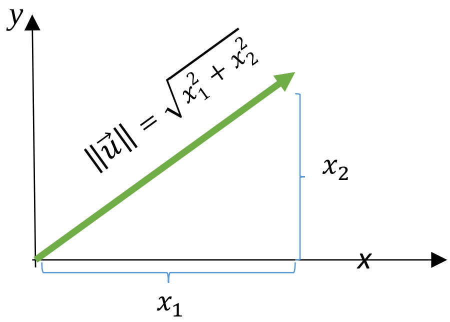
Similar formula holds in $\mathbb{R}^3$.

Norm of a Vector

Let $\vec{u} = \binom{x_1}{x_2}$ be a vector in the 2D plane, then its "length" or "norm" is given by the Pythagoras' Theorem by:

$$\sqrt{x_1^2 + x_2^2}$$

and is given the notation $\|\vec{u}\|$.



Remark

The word “norm” means the same thing as “length”. Similar formula holds in $\mathbb{R}^3$.

Inner product

Given any two vectors $\vec{u} = \binom{x_1}{x_2}$, $\vec{v} = \binom{y_1}{y_2}$

in 2D or 3D plane/space, their inner product is a “scalar” given by $x_1y_1 + x_2y_2$ and is denoted by (2D case):

$$\vec{u} \cdot \vec{v}$$

Hence

(1) $\quad \vec{u} \cdot \vec{v} = x_1y_1 + x_2y_2$

On the other hand, one can prove the formula

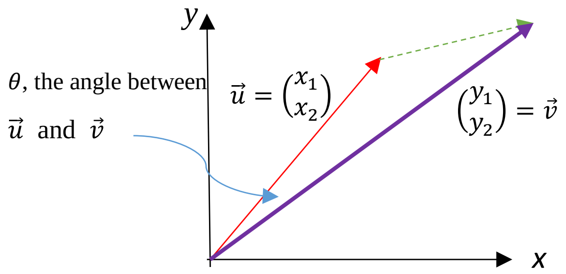
(2) $\quad \vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos\theta$

where $\|\vec{v}\| = \sqrt{y_1^2 + y_2^2}$

and θ is the angle between the vector $\vec{u}$ and $\vec{v}$.

Use of (1) and (2). Combining them, we can compute the angle between $\vec{u}$ and $\vec{v}$ (provided they both have non-zero norms).

This is done by the formula: $\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$



Remark

Similar formula holds in $\mathbb{R}^3$, i.e.

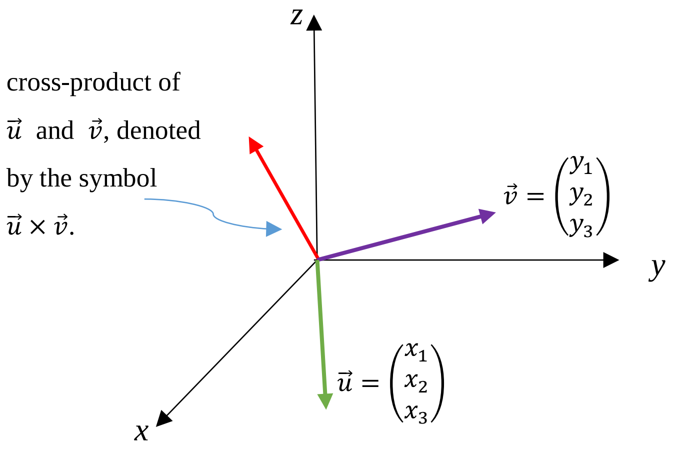
(1) becomes $\vec{u} \cdot \vec{v} = x_1 y_1 + x_2 y_2 + x_3 y_3$

(2) becomes $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta,$ where $\|\vec{u}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$
and $\|\vec{v}\| = \sqrt{y_1^2 + y_2^2 + y_3^2}$

Cross Product

We haven't mentioned this product in the lecture.

This kind of product produces "a vector" from "two vectors". It works only in $\mathbb{R}^3$.
The picture is:



Remark The cross product, i.e. $\vec{u} \times \vec{v}$, of the vectors $\vec{u}$ and $\vec{v}$ is the red vector which is perpendicular to both the vector $\vec{u}$ and the vector $\vec{v}$.

How to compute $\vec{u} \times \vec{v}$?

It uses the concept of 3×3 determinant (not mentioned in lecture! Optional).