MATH1010D (Wk2.1,2.2)

Keywords: Arithmetic of functions, limits of functions, arithmetic of limits, (monotone convergence theorem, uniqueness of limits, special limits)

Comment: Topics in brackets not yet discussed.

Arithmetic of Functions

When we have two functions, say f, g, both with the same (i) domain, (ii) codomain, then we can form new functions with the "names" $f+g, f-g, f\cdot g$ and f/g. They are defined by the rules:

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) + g(x)$$
$$(f\cdot g)(x) = f(x) + g(x)$$
$$(f/g)(x) = f(x) + g(x)$$

Remarks

1. Often times, people write fg for the function $f\cdot g$
2. The function f/g is only defined for those x satisfying the condition $g(x) \neq 0$.
3. The expression colored in "red" on the left-hand side are the "names" of each of these functions.
4. The right-hand sides denote the "rules" of obtaining the "value" of each of these functions at the point "x".

Using these rules, one can build new functions out of simple functions.

Limit of Functions

In calculus, one tries to understand functions. One question we are especially interested in answering is:

Question: given a function $f: A \to B$, is it 1-1, onto?

We ask this question, because we want to know

Question: Given $f: A \to B$,when does it have an inverse function?

- To show 1-1, one needs to show "whenever $f(x_1) = f(x_2)$ $then$ $x_1 = x_2$."
- To show onto, one has to determine for <u>which y</u> does the equation $y = f(x)$ has solution(s).

Both of these two are not too easy to answer. For example, you can try to use the methods outline in the 2 bullet points above to show that

Example The function $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^{2n+1}$, $n = 1,2,3,\cdots$ is 1-1 and onto.

An Easier Approach

We need the concept of "limit" and of "derivative".

For example, we will show (in the next lectures) that

- If $f'(x) > 0\ \forall x \in Dom$ then f is 1-1 (Similar for $f'(x) < 0$)
- If $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = -\infty$, and f is continuous at every point, then $f: \mathbb{R} \to \mathbb{R}$ is onto. (Many other cases exist, but we'll not discuss them here!)

A Concrete and Simple Example

$$f: (0, \infty) \to (0, \infty)$$

given by $f(x) = x^3$ is 1-1 and onto.

How would you do it?

(Method from definition).

To show 1-1, show "$\forall x_1, x_2 \in (0, \infty): f(x_1) = f(x_2) \Longrightarrow x_1 = x_2$."

To do this, we argue this way: $x_1^3 = x_2^3 \Longrightarrow x_1^3 - x_2^3 = 0$

$$\Longrightarrow (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = 0$$

But the second factor on the right-hand side cannot be zero for any real nos. x_1, x_2, so $x_1 = x_2$.

To show onto, solve the equation $y = x^3$ for each y in $(0, \infty)$.

To do this, we just use the fact that $x = \sqrt[3]{y}$ has a solution for each real no. y.

Can we do it in another way? What about if we change the function to $f(x) = x^{2n+1}, n \in \mathbb{N}$?

As mentioned above, tools like (i) derivative, (ii) limit will be useful in showing 1-1 and ontoness of a function. Now we introduce these concepts. First we have

Limit

We start with two obvious limits, using them we can build up more complicated limits from them using the "arithmetic" of limits.

Two Obvious Limits

1. $\lim\limits_{x\to\infty} x = \infty.$

2. $\lim\limits_{x\to\infty} \frac{1}{x} = 0^+.$

   (here 0^+ is the notation for "limit is approached from positive nos. to zero").

Arithmetic of Limits

Given two functions, both of them having limits, we can compute the "limit" of the "sum, difference, product or quotients" of them as follows:

1. $\lim\limits_{x\to\infty} (f \pm g)(x) = \lim\limits_{x\to\infty} f(x) \pm \lim\limits_{x\to\infty} g(x)$.
2. $\lim\limits_{x\to\infty} (f \cdot g)(x) = \lim\limits_{x\to\infty} f(x) \cdot \lim\limits_{x\to\infty} g(x)$.
3. $\lim\limits_{x\to\infty} (f/g)(x) = \lim\limits_{x\to\infty} f(x) / \lim\limits_{x\to\infty} g(x)$.

Remark

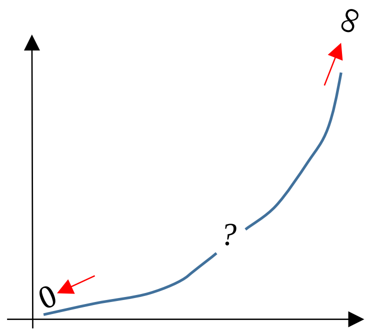
These lines should be understood as: "if the limit on the right-hand side (exists) and is (finite), then the limit on the left-hand side (exists) and is (equal to) the right-hand side".

Using Limit to show Onto-ness for $f(x) = x^3$ with domain $(0, \infty)$.

We need something we didn't mention but is obvious.

This is: $\lim\limits_{x\to\infty} x^3 = \infty$ and $\lim\limits_{x\to 0^+} x^3 = 0^+$.

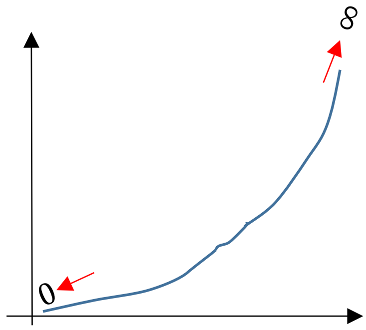
Using this, we see that the function looks something like:



Remarks

- When $x \to 0^+$, the function values $f(x) \to 0$ (to be more precise, 0^+).
- When $x \to \infty$, the function values $f(x) \to \infty$.
- We don't know what happens at the place where we put a question mark!
- In the next lectures, we will discuss the concept of "continuity" and a theorem (the Intermediate Value Theorem). They together will guarantee that the function $f(x) = x^3$ defines a connected "curve". Because of this, the function's graph (i.e.

the “curve” looks like)



(Intuitively, the Intermediate Value Theorem says: “each value between the maximum and the minimum values of f will be achieved.”)

One more comment:

One also need to ensure that $f(x) > 0, \forall x \in (0, \infty)$.

If we can show all the above, then we know that the range of the function $f: (0, \infty) \to (0, \infty)$ given by $f(x) = x^3$ is onto.

More Limit Rules

There are also other obvious limits, such as

Some Other Obvious Limits

1. $\lim_{x \to 0} x = 0$. (meaning $x \to 0^+$ as well as $x \to 0^-$)
2. $\lim_{x \to 0^+} \frac{1}{x} = \infty$. (similarly, $\lim_{x \to 0^-} \frac{1}{x} = -\infty$.)

We also have

3. $\lim_{x \to c} x = c$. (meaning $x \to c^+$ as well as $x \to c^-$)

The following are some more rules for $+, -, \times, \div$ of limits. They can be proved using “epsilon-delta definition” (a complicated definition). We will not prove them. You can assume that these rules are true.

Arithmetic of Limits

Given two functions, both of them having limits, we can compute the “limit” of the “sum, difference, product or quotients” of them as follows:

1. $\lim_{x \to c} (f \pm g)(x) = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$.
2. $\lim_{x \to c} (f \cdot g)(x) = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$.
3. $\lim_{x \to c} (f/g)(x) = \lim_{x \to c} f(x) / \lim_{x \to c} g(x)$.

provided that $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, where L, M are finite numbers.

Remark

In case of division, i.e. item 3, we have to make the extra assumption that $\lim_{x \to c} g(x) \neq 0$.

A Worked Example to show onto-ness of a function w/o (= without) solving equation

> **Example**
>
> Show that the function $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ with domain $\mathbb{R}$ is an onto function to the set $(-1,1)$.
>
> **Suggested Solution Steps.**
>
> Step 1). Simplify the function to get $f(x) = \frac{e^x(1-e^{-2x})}{e^x(1+e^{-2x})} = \frac{1-e^{-2x}}{1+e^{-2x}}$
>
> Step 2) Find the limits of f when $x \to \pm\infty$.
>
> Step 3) Show that $-1 \leq f(x) \leq 1, \forall x$
>
> Question: Finish the proof yourself !

The following limits are especially useful.

Some Special Limits

1. $\lim_{x \to \infty} \frac{x}{e^x} = 0$
2. $\lim_{x \to 0} \frac{\sin x}{x} = 1$
3. $\lim_{x \to 0} \sin x = 0.$

The 3 limits mentioned above can be proved by "comparing" them to some well-known limits. One main tool for such comparison is:

Squeeze Theorem (or Comparison Theorem or Sandwich Theorem)

Suppose the functions $\hat{A}, \hat{B}, \hat{C}$ with the same domain satisfy $\hat{A}(x) \leq \hat{B}(x) \leq \hat{C}(x)$ for all x in domain of all 3 functions.

Furthermore, suppose $\lim_{x \to c} \hat{A}(x) = L = \lim_{x \to c} \hat{C}(x)$, then the function $\hat{B}(x)$ has also the limit L, i.e. $\lim_{x \to c} \hat{B}(x) = L$.

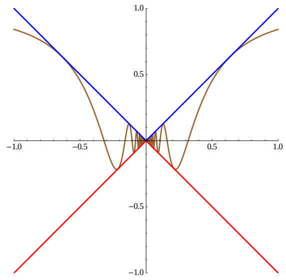
Remark

The point c doesn't need to be in the "domain"!

Examples of the use of Squeeze/Sandwich Theorem

Consider. $\lim_{x \to 0} x \cdot \sin \frac{1}{x}$. This function is not defined when $x = 0$.

Picture of this function



By Squeeze Theorem, we have

$$0 \le |x \cdot \sin\frac{1}{x}| \le |x|$$

So $\hat{A}(x) = 0 \ \forall x, \hat{B}(x) = x\sin\frac{1}{x}, \hat{C}(x) = |x|$ and the Squeeze Theorem gives (since $\lim_{x\to 0}\hat{A}(x) = 0$ and $\lim_{x\to 0}\hat{C}(x) = 0$ that $\lim_{x\to 0}\hat{B}(x) = 0$.

Conclusion: We have shown $\lim_{x\to 0} x\sin\frac{1}{x} = 0$.

The Special Limit $\lim_{x\to 0}\frac{\sin x}{x} = 1$

Suggested Solution (Two Ways: (i) via Pictures, (ii) via Infinite Polynomials (i.e. Power Series)

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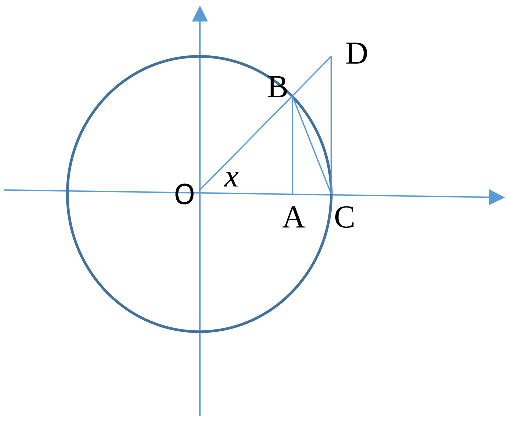

It is easy to check the area inequalities (Two vertical lines means "area of"):

$$|\Delta OBC| < |circular\ sector\ OBC| < |\Delta ODC|$$

This implies

$$\frac{1}{2}\sin x < \frac{1}{2}x < \frac{1}{2}\tan x = \frac{1}{2}\frac{\sin x}{\cos x}$$

After simplification, we obtain

$$\sin x < x < \frac{1}{\cos x}$$

implying

$$1 < \frac{x}{\sin x} < \cos x \Leftrightarrow 1 < \frac{\sin x}{x} < \frac{1}{\cos x}$$

Letting $x \to 0$, we obtain (using the Squeeze/Sandwich Theorem) the following:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

- (Via Power Series) This is a little subtle. The idea is:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = x(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots)$$

Now one tries to argue that $\lim_{x\to 0} 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots = 1.$

> This step is not too easy, as it is about an "infinite sum". The main idea is to compare it with some geometric series (i.e. something like $a + a^2 + a^3 + \cdots$)..

Makeup Class

Keywords: continuity, sequential limit, picture of cont./discontinuous function at a point c, the limit for e, a few words on the vocabulary "asymptotes"

One Notation

When we write

$$\lim_{x\to c} f(x) = L$$

we mean

- Left-limit, i.e. $\lim_{x\to c^-} f(x) = L_1$ exists (and is finite)
- Right-limit, i.e. $\lim_{x\to c^+} f(x) = L_2$ exists (and is finite)
- $L_1 = L_2$.

Continuity at $x = c$.

If we add the condition "$f(c) = L_1 = L_2$" to the above 3 bullet point, we get

- $\lim_{x\to c^-} f(x) = L_1, \ \lim_{x\to c^+} f(x) = L_2, \ L_1 = L_2, \ f(c) = L_1 = L_2$

In such a case, we say that "f is continuous at the point c".

Example

Consider the function $f(x) = \begin{cases} \frac{x^2-1}{x-1} & x \neq 1 \\ -2 & x = 1 \end{cases}$

Then we can check that

- $\lim_{x \to 1^-} f(x) = 2$
- $\lim_{x \to 1^+} f(x) = 2$
- $f(1) = -2.$

Therefore f is <u>not</u> continuous at the point $1.$

Remark

If we redefine f so that $f(1) = 2,$ then the function is continuous at $2.$

Sequential Limit and Discontinuity at a point

One convenient tool to show "discontinuity" at a point c is to use "sequential limit". The idea is this.

The expression $\lim_{x \to c} f(x) = L$ means "no matter how <u>x approach c</u>, $f(x)$ will approach L accordingly."

We can then describe how x approach c by considering x to be a sequence of numbers, indexed by $n.$ Then, the phrase "x approach c" becomes "$x_n \to c$" (or $\lim_{n \to \infty} x_n = c$).

Next, each of these x_n gives rise to a value of the function which is $f(x_n)$. So the set $\{f(x_n)\}$ is also a sequence of numbers. This sequence satisfies

$$x_n \to c \; then \; f(x_n) \to L$$

Or simply $\lim_{n \to \infty} f(x_n) = L.$

Application

Sequential limit is especially useful, when you need to show "no limit" at $c.$ You just choose two different sequences, say $\{x_n\}, \{z_n\}$, both approaching $c,$ such that

$$\lim_{n \to \infty} f(x_n) = L$$

$$\lim_{n \to \infty} f(z_n) = M$$

and

$$L \neq M.$$