MATH1510-i
Proofs of Differentiation Rules

Covered:

- Four rules of derivatives (i.e. $+, -, \times, \div$)
- Briefly Mention Chain Rule (i.e. derivative of composite function of two functions)

Four rules of derivatives

Assumption: In the following let $f(x), g(x)$ be two functions, both having the same domain, and both differentiable at the point $x = c$ in the domain. Then we have (*) $f(x) \pm g(x)$, $f(x)g(x)$, $f(x)/g(x)$ are all differentiable at $x = c$. (For the last one, one has to make the extra assumption that $g(c) \neq 0$.)
Furthermore, the derivatives of these "sum", "difference", "product" and "quotient" functions at the point $x = c$ are given by formulas listed below:

1. The derivative of the sum function $f(x) + g(x)$ at $x = c$ (If you like, you can give a name to this function, calling it for example $(f + g)(x)$ or $h(x)$) has the following formula.
$$\left.\frac{d(f(x) + g(x))}{dx}\right|_{x=c} = f'(c) + g'(c)$$
2. Similarly, for the function $f(x) - g(x)$, we have
$$\left.\frac{d\big(f(x) - g(x)\big)}{dx}\right|_{x=c} = f'(c) - g'(c)$$
3. (Product Rule) For product of these two functions, the formula is slightly different, i.e.
$$\left.\frac{d(f(x)g(x))}{dx}\right|_{x=c} = g(c)f'(c) + g'(c)f(c)$$
Remark: In the case when $g(x) \equiv k$ (i.e. it is constantly equal to k), the above formula has the simpler form
$$\left.\frac{d(kf(x))}{dx}\right|_{x=c} = kf'(c)$$
4. (Quotient Rule) For quotient, it is
$$\left.\frac{d\big(f(x)g(x)\big)}{dx}\right|_{x=c} = \frac{f'(c)g(c) - g'(c)f(c)}{\big(g(c)\big)^2}$$

Idea of Proof of the Product Rule

We just outline one or two of the ideas. If you are interested in more detail, just send me an e-mail. I will explain more to you.

A Preparatory Theorem

To show the product rule, we need the following "little" result (called "lemma"):

Lemma (Differentiable at $x = c$ $\Rightarrow$ continuous at $x = c$.)

Assume $f(x)$ is differentiable at $x = c$, then $f(x)$ is continuous at $x = c$.

Proof:

Main idea is to start from the statement $\lim_{h\to 0} f(c+h) = f(c)$ (definition of "continuous at $x = c$.") and try to connect it to i.e. $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = f'(c)$ (definition of "differentiable at $x = c$.").

The connection can be established if one looks at the expressions:

(i) $f(c+h) - f(c)$ and

(ii) $\frac{f(c+h)-f(c)}{h}$

This is because $f(c+h) - f(c) = \frac{\left(f(c+h)-f(c)\right)}{h} \cdot h$

Now we know that in the above equation, both of the limits $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ and the limit $\lim_{h\to 0} h$ exist.

Furthermore, the first limit (i.e. $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$) is equal to $f'(c)$, which is a finite number. The second limit (i.e. $\lim_{h\to 0} h$) is equal to zero.

Combining all these, we get for the right-hand side:

$$\lim_{h\to 0} \frac{\left(f(c+h)-f(c)\right)}{h} \cdot \lim_{h\to 0} h = f'(c) \cdot 0 = 0$$

It follows that the limit of the left-hand side also exists and is given by

$$\lim_{h\to 0}\big(f(c+h)-f(c)\big)=\lim_{h\to 0}\left(\frac{\big(f(c+h)-f(c)\big)}{h}h\right)=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}\cdot\lim_{h\to 0}h$$
$$=0$$

Steps of the Proof of Product Rule

1. Consider the "Difference Quotient" i.e.

$$\frac{f(c+h)g(c+h)-f(c)g(c)}{h}$$

2. Rewrite it in the form (because we only know the following limits to exist: (i) $\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$, (ii) $\lim_{h\to 0}\frac{g(c+h)-g(c)}{h}$):

$$\frac{f(c+h)g(c+h)-f(c+h)g(c)+f(c+h)g(c)-f(c)g(c)}{h}$$

Grouping terms we get from the above:

$$\frac{f(c+h)[g(c+h)-g(c)]}{h}+g(c)\,\frac{f(c+h)-f(c)}{h}$$

3. Take limit $h\to 0$. The term $\frac{g(c+h)-g(c)}{h}$ goes to the limit $g'(c)$. On the other hand, the term $\frac{f(c+h)-f(c)}{h}$ goes to the limit $f'(c)$. (You can write these two facts in the form: $\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}=f'(c)$ and $\lim_{h\to 0}\frac{g(c+h)-g(c)}{h}=g'(c)$).

4. We still have two more limits to consider. They are:

   (i) $\lim_{h\to 0}f(c+h)$ and

   (ii) $\lim_{h\to 0}g(c)$.

   Since $f(x)$ is <u>differentiable</u> at $x=c$, it is <u>continuous</u> at $x=c$. So the first one is just $\lim_{h\to 0}f(c+h)=f(c)$. As for the second one, $g(c)$ is a constant function, so its limit is given by $\lim_{h\to 0}g(c)=g(c)$.

5. Combining all the above, we get

$$\lim_{h\to 0}f(c+h)\,\lim_{h\to 0}\frac{[g(c+h)-g(c)]}{h},+g(c)\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

$$= f(c)g'(c) + g(c)f'(c).$$

Appendix

I didn't give proofs of the Quotient Rule and the Chain Rule.

Here below are some ideas about their proofs. If you're interested, you can take a look at them.

Quotient Rule

This rule is: $\left(\frac{f}{g}\right)'(c) = \frac{g(c)f'(c)-f(c)g'(c)}{[g(c)]^2}$, where $g'(c) \neq 0$

Proof:

Step 1: Prove the simple case $\left(\frac{1}{g}\right)'(c) = \frac{-g'(c)}{[g(c)]^2}$

To prove this, consider $\frac{\left(\frac{1}{g(x)}-\frac{1}{g(c)}\right)}{x-c} = \frac{-(g(x)-g(c))}{(x-c)g(x)g(c)}$

Step 2: Let $x \to c$.

***** (The difficulty)** Make sure that "from the assumption $g(c) \neq 0$, it follows that for all x near to c, $g(x) \neq 0$ also. (This requires more mathematics and can be done using $\varepsilon - \delta$ language).

Assuming that this can be proved, we then have

$$\lim_{x\to c}\frac{\left(\frac{1}{g(x)}-\frac{1}{g(c)}\right)}{x-c} = \lim_{x\to c}\frac{-(g(x)-g(c))}{(x-c)g(x)g(c)} = \frac{1}{g(c)}\cdot\lim_{x\to c}-\frac{g(x)-g(c)}{x-c}\lim_{x\to c}\frac{1}{g(x)}$$

$$= \frac{1}{g(c)}\cdot\lim_{x\to c}-\frac{g(x)-g(c)}{x-c}\cdot\frac{1}{\lim_{x\to c}g(x)} = \frac{1}{[g(c)]^2}g'(c)$$

as required.

Step 3: Next we use the product rule to get $\left(\frac{f}{g}\right)'(c) = \frac{g(c)f'(c)-f(c)g'(c)}{[g(c)]^2}$.

To do this, we consider $\left(f\cdot\left(\frac{1}{g}\right)\right)'(c) = f(c)\cdot\left(\frac{1}{g}\right)'(c) + f'(c)\cdot\left(\frac{1}{g(c)}\right)$

$$= f(c)\cdot\left(-\frac{g'(c)}{\left(g(c)\right)^2}\right) + \frac{f'(c)}{g(c)}$$

$$= \frac{-f(c)g'(c) + g(c)f'(c)}{\left(g(c)\right)^2}$$

as required.

Chain Rule

Intuitive Idea:

This rule is like "cancellation" of fractions, e.g. $\frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$.

For derivatives, we have $\frac{df(u),}{du} \cdot \frac{du(x)}{dx} = \frac{df(u(x))}{dx}$

Example. $\frac{d \sin(e^x)}{dx} = \frac{d \sin(u)}{du} \frac{du}{dx}$, where we let $u = e^x$.

Now $\frac{d \sin(u)}{du} = \cos(u)$, but $u = e^x$, so $\frac{d \sin(u)}{du} = \cos(e^x)$.

Next, $\frac{du}{dx} = \frac{d\ e^x}{dx} = e^x$.

Combining everything, we get $\frac{d \sin(e^x)}{dx} = \cos(e^x)e^x$

Remark:

In the line above, I sometimes wrote u, at other times $u(x)$. Both mean the same thing – i.e. it is about the function u which is a function of the variable x. When I wrote u, I am omitting the variable x to make the notation simpler. When I wrote $u(x)$, I want to make the reader understand that this function, u, depends on x.

Another way of writing the Chain Rule:

$$f\left(g(c)\right)' = f'\left(g(c)\right) \cdot g'(c)$$

Proof Idea:

To prove it, first consider $\frac{f\left(g(c+h)\right) - f\left(g(c)\right)}{h} = \frac{f\left(g(c+h)\right) - f\left(g(c)\right)}{g(c+h) - g(c)} \cdot \frac{g(c+h) - g(c)}{h}$

Then let $g(c + h) = g(c) + k$, this leads to "as $h \to 0$, then $k \to 0$" by the continuity of $g(x)$ at $x = c$.

Hence $\frac{f(g(c+h))-f(g(c))}{g(c+h)-g(c)} \cdot \frac{g(c+h)-g(c)}{h} = \frac{f(g(c)+k)-f(g(c))}{k} \cdot \frac{g(c+h)-g(c)}{h}$

and also $\lim_{h\to 0} \frac{f(g(c+h))-f(g(c))}{g(c+h)-g(c)} \cdot \frac{g(c+h)-g(c)}{h} = \lim_{h\to 0} \frac{f(g(c)+k)-f(g(c))}{k} \cdot \lim_{h\to 0} \frac{g(c+h)-g(c)}{h}$

$$= \lim_{k\to 0} \frac{f(g(c)+k)-f(g(c))}{k} \cdot \lim_{h\to 0} \frac{g(c+h)-g(c)}{h}$$
$$= f'(g(c)) \cdot g'(c)$$

Remark: Question is: "What happens if $g(c+h)-g(c)=0$?"

Answer: Do directly the computation, i.e.

$$\lim_{h\to 0} \frac{f(g(c+h))-f(g(c))}{h} = \lim_{h\to 0} \frac{f(g(c)+k)-f(g(c))}{k} \lim_{h\to 0} \frac{g(c+h)-g(c)}{h}$$

Here though k may be zero, but it doesn't matter. Why? Because now the left-hand side is zero, while the left-hand side contains 2 terms, the term $\lim_{h\to 0} \frac{f(g(c)+k)-f(g(c))}{k}$ and the term $\lim_{h\to 0} \frac{g(c+h)-g(c)}{h}$. Both of them are known to be finite numbers (because we are assuming that the derivative $f'(g(c))$ and $g'(c)$ exists.

Now one of them is zero, i.e. $\lim_{h\to 0} \frac{g(c+h)-g(c)}{h} = 0$ because $g(c+h)=g(c)$ for infinitely many values of h near 0. It follows that $g'(c)=0$.

Finally, we know that $f'(g(c)) \cdot g'(c) = a\ finite\ number \times 0 = 0$.

So left-hand side (which is zero) is equal to the right-hand side, which is also zero.

Implicit Differentiation

In high schools, you may have learned this way of computing derivative of a function:

$$x^2 + y^2 = a^2$$

Then compute the derivative of x, then of y, then of a (which is on the right-hand side of the equation) all with respect to the independent variable x. Having done this, we obtain

$$\frac{dx^2}{dx} + \frac{dy^2}{dx} = \frac{da^2}{dx}$$

Now $\frac{dx^2}{dx} = 2x$, $\frac{dy^2}{dx} = \frac{dy^2}{dy}\frac{dy}{dx} = 2y \cdot y'$ and $\frac{da^2}{dx} = 0$

Result: We get now $2x + 2yy' = 0$ implying $y' = \frac{-x}{y}$.

Remark: To compute the value of this derivative, we need two numbers, i.e. both x and y. Or we can express y in terms of x using the equation

$$x^2 + y^2 = a^2$$

to get $y' = -\frac{x}{\pm\sqrt{a^2-x^2}} = \mp\left(\frac{x}{\sqrt{a^2-x^2}}\right)$.

Question: Why can we do this?

Answer: This is due to the

Implicit Function Theorem, which roughly says:
Given any function of two variables x and y, i.e. $f(x,y)$ and an equation $f(x,y) = c$ (the right-hand side is a constant), then we have

1. y is a function of x or
2. x is a function of y.

In symbols, we write the sentence "y is a function of x" as "$y = y(x)$". (We don't write things like "$y = f(x)$" because that would need an extra letter f.)
Similarly, the second sentence becomes $x = x(y)$.

Examples:

1. Find $\frac{d\ln x}{dx}$.

   Answer: Let $y = \ln x$, then $e^y = x$ or $e^y - x = 0$. Now we have an equation of the form $F(x,y) = 0$.
   By the Implicit Function Theorem, we know that "implicitly" y is a function of x. (symbolically written as $y = y(x)$)
   So we can differentiate both sides of the equation $e^y - x = 0$ with respect to

x to obtain $\frac{d}{dx}(e^y - x) = \frac{d0}{dx} \Longrightarrow \frac{de^y}{dx} - \frac{dx}{dx} = 0 \Longrightarrow \frac{de^y}{dy}\frac{dy}{dx} - 1 = 0$

$\Longrightarrow e^y \cdot \frac{dy}{dx} = 1 \Longrightarrow \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$ (because by definition $e^y = x$).

2. Find $\frac{d\,\mathrm{arcsin}(x)}{dx}$.

   Answer: Let $y = \arcsin(x)$, then $\sin(y) = x$. Hence $\sin(y) - x = 0$. Differentiate both sides of the equation (remembering that $y = y(x)$). we obtain $\frac{d}{dx}(\sin(y) - x) = \frac{d0}{dx} \Longrightarrow \frac{d\sin(y)}{dy}\frac{dy}{dx} - \frac{dx}{dx} = 0 \Longrightarrow \cos(y)y' = 1$

$$y' = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1 - \sin^2(y)}} = \frac{1}{\sqrt{1 - x^2}}$$

Remark: We said "roughly" because the theorem requires some "differentiability" conditions on the function $f(x, y)$ which is usually satisfied. Also, the "or" can mean "either/or" or "both".