

Application 1

- Thm 3 Let
- $M =$ complete simply-connected Riem. mfd with
 - $K \leq 0$ (sectional curvature)
 - $O \in M$ is a fixed point.
 - $\rho: M \rightarrow [0, \infty)$ (the distance function wrt O) is defined by

$$\rho(x) = d(x, O).$$

Then $\rho^2 \in C^\infty(M)$ and $D^2 \rho^2 > 0$

(strictly positive definite)

Ex: If $M = \mathbb{R}^n$, $O = \text{origin}$, then $\rho^2(x) = |x|^2$ &
 $D^2\rho^2(v, v) = c|v|^2$ (for some $c > 0$)

Pf of Thm 3: By Cartan-Hadamard Thm,

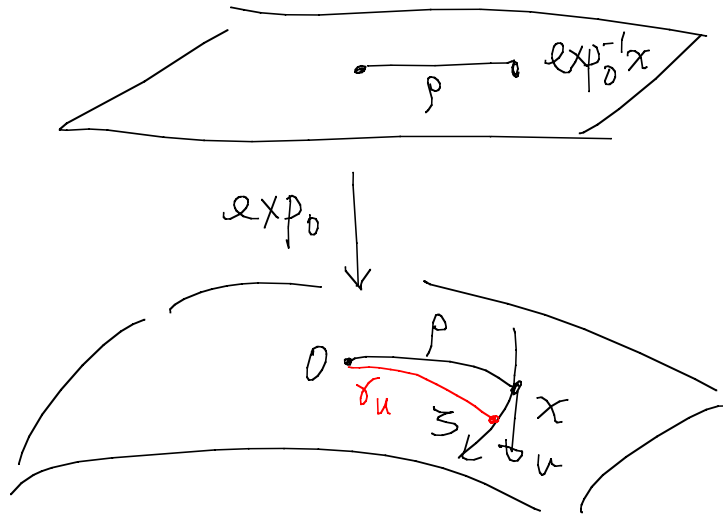
$$\rho(x) = |(\exp_{p_0})^{-1}(x)|,$$

Therefore $\rho^2(x) = |(\exp_{p_0})^{-1}(x)|^2$ is smooth.

Now suppose $x \neq 0$, and $v \in T_x M$

Take a curve $\zeta: [-\varepsilon, \varepsilon] \rightarrow M$ s.t.,

$$\zeta(0) = x \quad \& \quad \zeta'(0) = v.$$



For each $u \in [-\varepsilon, \varepsilon]$, let $\gamma_u = [0, b] \rightarrow M$ (with $b = \rho(x)$)
 is the unique geodesic joining 0 to $\zeta(u)$. Note
 that $\gamma_0 = \gamma = [0, b] \rightarrow M$ is a normal geodesic
 (other γ_u may not be a normal geodesic).

In particular, we can choose $\zeta(u)$ to be a geodesic.

Then, the end point of γ_u is $\gamma_u(b) = \zeta(u)$

\Rightarrow the transversal vector field $V(x, u)$ at $x=b$

$$\bar{v} \quad V(b, u) = \zeta'(u).$$

Therefore $D_u V \Big|_{(b, u)} = D_{\zeta'(u)} \zeta'(u) = 0$ since $\zeta = \text{geodesic}$.

Also, at $x=0$, we have $\gamma_u(0) = 0$ & $V(0, u) \equiv 0$

$$\Rightarrow D_u V \Big|_{(0, u)} = 0.$$

Hence, the 2nd variation formula gives

$$\frac{d^2 L}{du^2}(0) = \int_a^b \left\{ |D_{\gamma'} \dot{\gamma}^\perp|^2 - \langle R_{\dot{\gamma}^\perp \gamma'} \dot{\gamma}^\perp, \gamma' \rangle \right\} dt$$

$$\geq \int_a^b |D_{\gamma'} \dot{\gamma}^\perp|^2 \quad (\text{since } K \leq 0).$$

Now $D^2 \rho^2(u, u) = \left\{ \zeta'(\zeta' \rho^2) - (D_{\zeta'} \zeta') \rho^2 \right\} \Big|_{u=0}$

$$= \zeta'(\zeta' \rho^2) \Big|_{u=0} \quad (\text{since } \zeta = \text{geodesic})$$

$$= \zeta'(2\rho \zeta' \rho) \Big|_{u=0}$$

$$= \left[2\rho \zeta'(\zeta' \rho) + 2(\zeta' \rho)^2 \right] \Big|_{u=0}$$

$$= 2\rho(x) \frac{d^2}{du^2} \Big|_{u=0} \rho(\zeta(u)) + 2 \left[\frac{d}{du} \Big|_{u=0} \rho(\zeta(u)) \right]^2$$

Note that $\rho(\zeta(u)) = L(\gamma_u) = L(u)$

$$\therefore \left. \frac{d}{du} \right|_{u=0} \rho(\zeta(u)) = \left. \frac{dL}{du} \right|_{u=0}$$

$$= \langle \gamma'(t), \psi(t) \rangle \Big|_0^b - \int_0^b \langle D_{\gamma'} \gamma', \psi \rangle dt$$

$$= + \langle \gamma'(b), \psi(b) \rangle \quad \left(\begin{array}{l} \text{since } \gamma = \text{geodesic} \\ \& \psi(0) = 0 \end{array} \right)$$

$$= + \langle \gamma'(b), \zeta'(0) \rangle$$

$$= + \langle \gamma'(b), \nu \rangle$$

$$\& \left. \frac{d^2}{du^2} \right|_{u=0} \rho(\zeta(u)) = \left. \frac{d^2 L}{du^2} \right|_{u=0} \geq \int_0^b |D_{\gamma'} \psi^\perp|^2 dt$$

$$\therefore D^2 \rho(\nu, \nu) \geq 2\rho(x) \int_0^b |D_{\gamma'} \psi^\perp|^2 dt + 2[\langle \gamma'(b), \nu \rangle]^2$$

If $\langle \gamma'(b), v \rangle \neq 0$, then $D^2 \rho^2(v, v) > 0$

If $\langle \gamma'(b), v \rangle = 0$, then $U(b) = v \in [\gamma'(b)]^\perp$

Note that $\{\gamma_u\}$ is a 1-param. family of geodesics,

U is a Jacobi field along γ . Hence

$$\langle \gamma'(b), U(b) \rangle = \langle \gamma'(0), U(0) \rangle = 0$$

$$\Rightarrow U(b) = U \neq 0$$

$\Rightarrow U(\star)$ is a nontrivial normal Jacobi field

$$\therefore U^\perp(\star) = U(\star).$$

Therefore $D_\gamma U^\perp = D_\gamma U \neq 0$. Otherwise, U is a

parallel transport of $\psi(0)=0 \Rightarrow \psi \equiv 0$ which is a contradiction. Hence

$$D^2\rho^2(u,u) \geq \int_0^b |D_\sigma \psi^\perp|^2 dt > 0.$$

This completes the proof of the thm. ~~#~~

The key point of the conclusion of the above thm is that

$D^2\rho^2 > 0$ on the whole M , which needs the curvature assumption. Otherwise, we have

Lemma 4 Let $\bullet M = \text{Riem mfd}$

- $0 \in M$

- $\rho = M \rightarrow \mathbb{R}$ distance to 0 .

Then \exists a nbd \mathcal{U}_0 of 0 in M s.t.

$$\rho^2 \text{ is smooth and } D^2\rho^2 > 0 \text{ in } \mathcal{U}_0.$$

PF: Let \mathcal{U} be a nbd of 0 s.t. \exists normal coordinate system $\{x^1, \dots, x^n\}$ centered at 0 . Using

this one can show that $\forall v, w \in T_0M$,

$$D^2\rho^2(v, w) = 2\langle v, w \rangle \quad (\text{Fx!})$$

Therefore, at the center 0 , $D^2\rho^2 > 0$.

$\Rightarrow D^2\rho^2 > 0$ in a nbd $\mathcal{U}_0 \subset \mathcal{U}$ of 0 . $\#$

Recall: • A function $f: M \rightarrow \mathbb{R}$ ($M = \text{Riem. mfd}$)
is said to be convex (strictly convex)

$\Leftrightarrow \forall$ geodesic γ in M ,

$f \circ \gamma$ is convex (strictly convex)

• Therefore, a C^∞ $f: M \rightarrow \mathbb{R}$ is convex (strictly convex) $\Leftrightarrow D^2 f \geq 0$ (> 0) (Ex!)

Def: Let $M = \text{complete Riem. mfd}$. Then

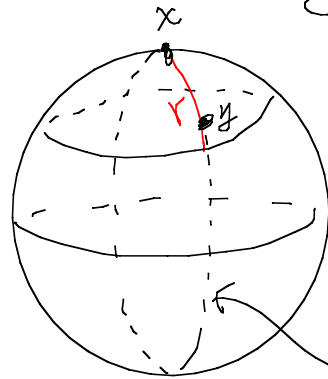
• a subset $\Omega \subset M$ is called convex

$\Leftrightarrow \forall x, y \in \Omega$, the shortest geodesic joining

x to y is contained in Ω .

- a subset $\Omega \subset M$ is called totally convex
 $\Leftrightarrow \forall x, y \in \Omega$, any geodesic joining x to y
is contained in Ω .

Eg 1: On $S^2 \subset \mathbb{R}^3$, geodesic ball of radius $r \leq \frac{\pi}{2}$
is convex, but not totally convex:

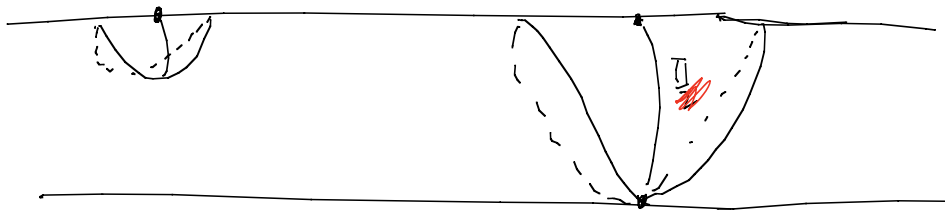


geodesic $\notin B_r(x)$
joining x to y .

Furthermore, geodesic ball of radius r between $\frac{\pi}{2}$ & π ,
is not even convex. (Ex!)

Note: If M is a simply-connected complete Riem. mfd
with nonpositive sectional curvature. Then
Cartan-Hadamard \Rightarrow any geodesic is minimizing.
Therefore, a convex subset of M is also totally
convex.

Eg 2 Cylinder $\{x^2 + y^2 = 1\} \subset \mathbb{R}^3$. Then B_r is convex
for $r \leq \frac{\pi}{2}$, not convex for $r > \frac{\pi}{2}$:



Lemma 5 Let $M = \text{Riem mfd}$

(1) Let $\tau: M \rightarrow \mathbb{R}$ is a convex function

• $M_c \stackrel{\text{def}}{=} \{x \in M : \tau(x) < c\}$ be the sublevel set.

• $\gamma: [a, b] \rightarrow M$ be a geodesic.

Then $\gamma(a), \gamma(b) \in M_c \Rightarrow \gamma([a, b]) \subset M_c$.

(2) Furthermore, if M is complete, then M_c is totally convex.

Pf: (1) $\tau \circ \gamma(t) \leq \max \{ \tau \circ \gamma(a), \tau \circ \gamma(b) \} < C$
Since $\tau \circ \gamma$ is convex,

(2) Easily follows from (1). ~~XX~~

Cor (of Thm 3): Geodesic balls of a simply-connected
complete Riem. mfd M with nonpositive sectional curvature
are totally convex.

In particular, $\forall x \in M$, $\{x\}$ is totally convex.

Therefore, there is no nontrivial geodesic $\gamma: [a,b] \rightarrow M$
s.t. $\gamma(a) = \gamma(b) = x$.

Thm 6 (J.H.C. Whitehead) Let $M = \text{Riem. mfd.}$. Then

$\forall x \in M, \exists$ a convex nbd. of x .

Pf: $\forall x \in M$, Lemma 4 (& properties of \exp_x)

$\Rightarrow \exists \varepsilon > 0$ s.t.

$\cdot \exp_x = B(\varepsilon) \xrightarrow{\subset T_x M} B_\varepsilon(x) \subset M$ is a diffeomorphism

$\cdot B_\varepsilon(x) = \exp_x(B(\varepsilon))$ has compact closure in M
(note that M may not be complete)

$\cdot \rho^2$ is C^∞ & $D^2 \rho^2 > 0$ on $B_\varepsilon(x)$, where $\rho = \text{distance to } x$.

In fact, by choosing a smaller $\varepsilon > 0$, we can also assume that $\forall y \in B_\varepsilon(x)$, $\exp_y|_{B(\varepsilon)}$ is a diffeomorphism.

Let $\delta = \frac{\varepsilon}{3} > 0$ and consider the geodesic ball $B_\delta(x)$.

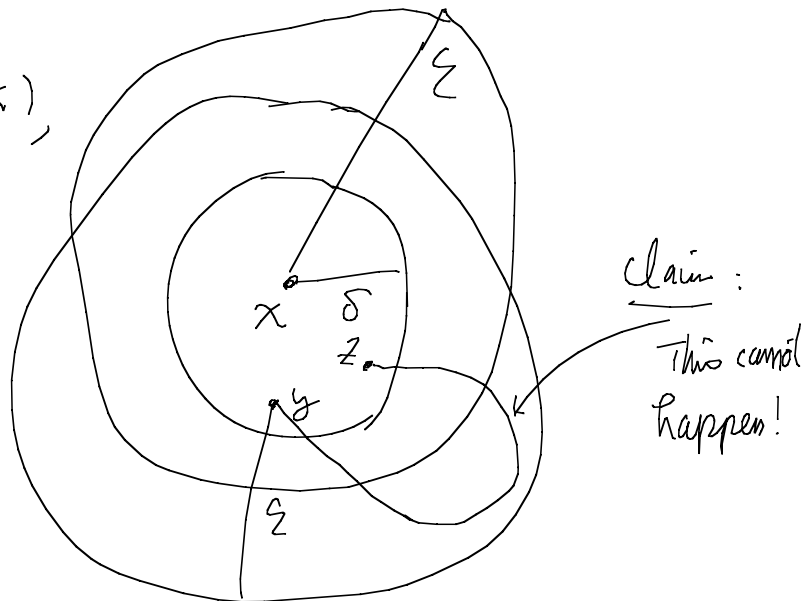
Then for a fixed $y \in B_\delta(x)$,

we claim that

$$B_\delta(x) \subset B_\varepsilon(y).$$

In fact, $\forall z \in B_\delta(x)$,

$$d(z, y) \leq d(z, x) + d(x, y)$$



$$\leq \delta + \delta = 2\delta = \frac{2\varepsilon}{3} < \varepsilon.$$

Therefore $\forall z \in B_\delta(x)$, \exists shortest geodesic ζ joining z to y with $\zeta \in B_\varepsilon(y)$ and $L(\zeta) < \varepsilon$.

Claim: $\zeta \subset B_\varepsilon(x)$.

Otherwise, $y, z \in B_\delta(x) \Rightarrow$

$$L(\zeta) > 2(\varepsilon - \delta) = 2\left(\varepsilon - \frac{\varepsilon}{3}\right) = \frac{4}{3}\varepsilon > \varepsilon$$

which is a contradiction.

Since $D^2\rho^2 > 0$ on $B_\varepsilon(x)$, statement (i) of Lemma 5 on

~~\Rightarrow the sublevel sets of ρ^2 are convex~~

$\Rightarrow B_\delta(x)$ is ~~convex~~ = sublevel set of ρ^2

$\Rightarrow \xi \subset B_\delta(x)$ since ξ is the shortest geodesic joining z to y .

Since $y \in B_\delta(x)$ is arbitrary, we've shown that

$\forall y, z \in B_\delta(x), \exists$ shortest geodesic $\xi \subset B_\delta(x)$ joining

y & z , $\therefore B_\delta(x)$ is convex. $\#$

Application 2: Synge Thm

~~Recall~~: • A C^∞ mfd M of n -dimension is said to be
Fact

orientable $\Leftrightarrow \exists$ a nowhere zero C^∞ n -form

ω on M . (i.e. $\omega = f dx^1 \wedge \dots \wedge dx^n$, $f > 0$; in local coordinates (x^1, \dots, x^n)
Alternating $(0, n)$ -tensor: $\omega(x_{i_1}, \dots, x_{i_2}, \dots, x_{i_n}) = -\omega(x_{i_2}, \dots, x_{i_1}, \dots, x_{i_n})$
 $\forall i < j$)

• If such an ω is chosen, then it is called the orientation of M . $(\omega_1 \sim \omega_2 \Leftrightarrow \omega_1 = f \omega_2)$
for some function $f > 0$

• Let ω be a nowhere zero n -form on such an M , then bases of $T_x M$ can be divided into 2 classes:

{
• positive oriented : $\omega(e_1, \dots, e_n) > 0$
• negative oriented : $\omega(e_1, \dots, e_n) < 0$ (wrt ω)

Lemma 7: Let $\gamma = [a, b] \rightarrow M$ be a closed curve in an orientable Riem. mfd M s.t. $x = \gamma(a) = \gamma(b)$.

Then the parallel transport along γ

$$P^\gamma: T_x M \rightarrow T_x M \text{ has } \det P^\gamma = +1.$$

Pf: (Ex!)

Lemma 8: Let $M =$ non simply-connected compact Riem. mfd. ($\pi_1(M) \neq 1$)

Then \exists closed curve $\gamma = [0, b] \rightarrow M$ (for some $b > 0$) such that $L(\gamma) \leq L(\alpha)$ for any piecewise

C^∞ closed curve α which is non-homotopic to zero (ie $\alpha \neq 1$).

(Pf = Omitted)

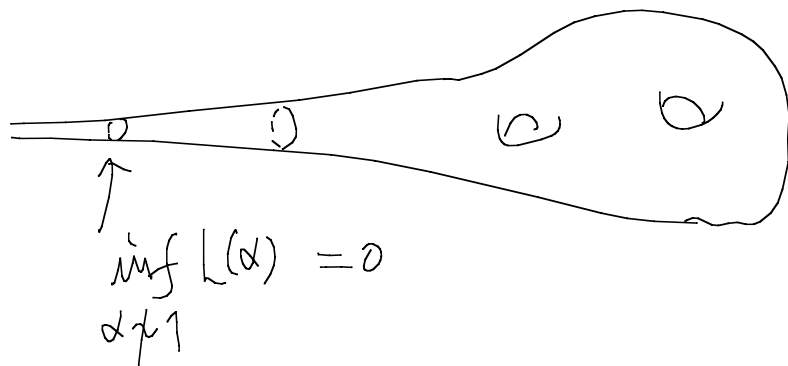
Note: • $\pi_1(M) \neq 1$ is necessary: any closed curve can be shrunk to a point

$$\Rightarrow \inf_{\gamma} L(\gamma) = 0$$

\Rightarrow no curve minimizing the length functional.

• compactness is also necessary:

eg: surface with cusp:



Thm 9 (J.L. Synge) If M is a compact orientable even
 dimensional Riem. mfd with positive sectional
 curvature, then M is simply-connected.

Pf: Suppose not, then $\pi_1(M) \neq 1$.

By Lemma 8, \exists a closed curve $\gamma = [0, b] \rightarrow M$ s.t.
 $L(\gamma) \leq L(\alpha)$, $\forall \alpha \neq 1$.

Then γ has to be a geodesic and hence

$$\gamma'(0) = \gamma'(b).$$

We may also assume $|\gamma'(t)| = 1$,

Let $x = \gamma(0) = \gamma(b)$. Then parallel transport

along γ :

$$P^\gamma = T_x M \rightarrow T_x M$$

has $\det P^\gamma = +1$.

Note that eigenvalues of P^γ are of the form ± 1 ,

$e^{i\theta}$ ($\neq \pm 1$), and if $e^{i\theta}$ is an eigenvalue, then

$e^{-i\theta}$ is also an eigenvalue.

Since $\det P^\gamma = +1$, the $\dim \{-1\text{-eigenspace}\}$ is even. Hence $\dim M = \text{even} \Rightarrow$

$\dim \{+1\text{-eigenspace}\}$ is also even,

Note that γ is a closed geodesic

$$\gamma'(0) = \gamma'(b) \quad \& \quad P^\gamma(\gamma'(0)) = \gamma'(b) = \gamma'(0)$$

$\Rightarrow \dim \{+1\text{-eigenspace}\} > 0$, hence ≥ 2 .

Therefore, $\exists e \in T_x M$ s.t. $P^\gamma(e) = e$,

$$\langle e, \gamma'(0) \rangle = 0$$

Now, let U be a parallel vector field along γ

$$\text{s.t. } U(0) = e.$$

$$\text{Then } U(b) = P^\gamma(U(0)) = P^\gamma(e) = e$$

$\therefore U$ is a well-defined vector field on the closed curve γ .

$\Rightarrow \exists$ a 1-parameter family of closed geodesics

$\{\gamma_u\}$ s.t. $\gamma_0 = \gamma$ & $U =$ transversal vector

field of $\{\gamma_u\}$. ($\gamma_u(t) = \exp_{\gamma(t)}(uU(t))$, $|u| \ll 1$)

Then, 2nd variation formula

$$\Rightarrow \frac{d^2 L}{du^2}(0) = \int_0^b [|D_\gamma U^\perp|^2 - \langle R_\gamma U^\perp \gamma', U^\perp \rangle] dt$$

(γ_u closed $\forall u \Rightarrow$ bdy term = 0)

Since $\langle U(0), \gamma'(0) \rangle = \langle e, \gamma'(0) \rangle = 0$, we have

$$U^\perp = U, \quad \forall t \in [0, b].$$

$$\Rightarrow D_\gamma U^\perp = D_\gamma U = 0 \quad (\text{since } U \text{ is parallel})$$

$$\Rightarrow \frac{d^2 L}{dt^2}(0) = - \int_0^b \langle R_{\gamma' U^\perp} \gamma', U^\perp \rangle dt$$

< 0 since sectional curvature > 0 .

Contradicting the assumption that γ is length minimizing. $\therefore \pi_1(M) = 1$ ~~✗~~