$\gamma(t) = \exp_x(tv)$ for some $v \in T_xM$.

Suppose that $I = (a_1, b_1)$ is the maximal possible interval containing $[0, \delta)$ s.t. $\gamma(t)$ is defined.

Suppose $b_1 < +\infty$. Then M complete $\Rightarrow$

$\exists\, y \in M$ s.t. $\lim_{t \to b_1} \gamma(t) = y$.

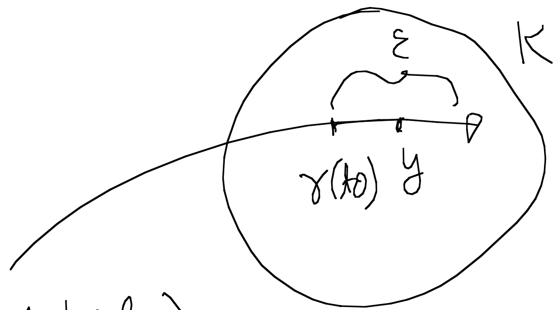
Let $K =$ cpt nbd. of y.

ODE theory $\Rightarrow$ $\exists\, \varepsilon > 0$ indep. of t_0 s.t.

" If $d(\gamma(t_0), y) < \frac{\varepsilon}{2}$, (s.t. $\gamma(t_0) \in K$)

Then $\exists$ normalized geodesic $\zeta : [0, \varepsilon] \to M$ s.t.

$$\zeta(0) = \gamma(t_0) \quad \& \quad \zeta'(0) = \gamma'(t_0)."$$



(Ex: check the detail)

$\Rightarrow$ joining ζ to γ gives an extension of γ beyond b_1.

Hence $b_1 = +\infty$.

Similar argument $\Rightarrow a_1 = -\infty$

$\therefore$ $\exp_x(tv)$ defined $\forall\, t \in (-\infty, \infty)$.

Since v is arbitrary, $\exp_x$ defined on whole $T_x M$.

(2) $\Rightarrow$ (3) trivial

(4) $\Rightarrow$ (1) is standard for metric space.

To prove (3) $\Rightarrow$ (4), we claim

(5) Assume $x \in M$ as in (3), then $\forall\, y \in M$, $\exists$ a minimizing geodesic joining x to y.

Pf of claim (5)

Let $\overline{B}(r) = \{ y \in M : d(x,y) \le r \}$

$$\Sigma(r) = \left\{ y \in \overline{\mathcal{B}}(r) : \begin{array}{l} y \text{ is joined to } x \text{ by} \\ \text{a min. geodesic.} \end{array} \right\}$$

Then we need to show

$$\overline{\mathcal{B}}(r) = \Sigma(r), \quad \forall\, r \in [0, \infty)$$

Let $J = \{ r \in [0,\infty) : \overline{\mathcal{B}}(r) = \Sigma(r) \}$

(1) Then we have already shown that

if $r < \delta$ where $\delta > 0$ is given by the "Thm" in the previous section,

$\exp_x$ diffeo

then $r \in J$

$\Longrightarrow J \neq \phi$.

(2) Next: since $\exp_x$ defined on whole $T_xM \cong \mathbb{R}^n$

continue dependence of $\exp_x(tv)$ on v

$\Rightarrow$ J is closed.

(3) To show J is open, we need the following fact

(Ex, or see do Carmo)

(*) $\forall$ cpt $K \subset M$, $\exists\, \varepsilon > 0$ s.t.

$\forall\, y, z \in K$ with $d(y,z) \le \varepsilon$,

then $\exists$ a minimizing geodesic joining y & z.

Note: This is a stronger result than the last Thm in §4.1

(in which one of the points has to be the center.)

<u>Pf of openness</u>: Define $K = \overline{\mathcal{B}}(2r)$, $\forall r$

Then $\overline{\mathcal{B}}(2r) \subset \exp_x(\overline{B}(2r))$

$\Rightarrow \overline{\mathcal{B}}(2r)$ cpt. (since $\overline{B}(2r)$ cpt in T_xM & $\exp_x$ diffeo.

$K \Leftarrow$

Applying $(*)$, $\exists\, \varepsilon > 0$ with property stated in $(*)$.

Let $\varepsilon' \in (0, \varepsilon)$ and $y \in \overline{\mathcal{B}}(r+\varepsilon')$.

If $y \in \overline{\mathcal{B}}(r)$, then $y \in \Sigma(r) \subset \Sigma(r+\varepsilon')$

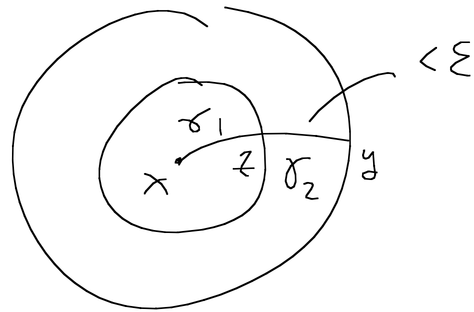
$(\because r \in S)$

If $y \in \overline{\mathcal{B}}(r+\varepsilon') \setminus \overline{\mathcal{B}}(r)$, then $\exists\, z \in \cancel{\partial \overline{B}(r)}\ \partial\overline{\mathcal{B}}(r)$

s.t. $\quad d(x,y) = d(x,z) + d(z,y)$

(by using cptness of $\partial \overline{B}(r)$ & definition of $d(x,y)$)

Then $r \in J \Rightarrow$



$\exists$ minimizing geodesic γ_1 joining x & z

On the other hand,

$$d(z,y) = d(x,y) - d(x,z) \le r + \varepsilon' - r = \varepsilon' < \varepsilon$$

$\overset{(*)}{\Longrightarrow}$ $\exists$ minimizing geodesic γ_2 joining z & y.

Then connecting γ_1 & γ_2, we have a (piecewise smooth) curve joining x & y with

length $= d(x,z) + d(z,y) = d(x,y)$

$\Rightarrow$ it must be a minimizing geodesic.

Therefore $\mathcal{J}$ is open.

Altogether, $\mathcal{J}$ is open, closed, nonempty subset of $[0,\infty) \Rightarrow \mathcal{J} = [0,\infty) \Rightarrow$ claim (5) ※

Pf of (3) $\Rightarrow$ (4)

By claim (5), $\forall$ bounded & closed set K,

$\exists A > 0$ s.t. $d(x,k) \leq A$, $\forall k \in K$

$\Rightarrow \quad K \subset \exp_x(\bar{B}(A))$

$\Rightarrow \quad K$ is cpt (since $\bar{B}(A)$ is cpt.)

$\#$

This completes the proof of Hopf-Rinow Thm.

<u>Pf of Cor 1</u>: Hopf-Rinow $\Rightarrow$ (2) is true

($\Rightarrow$ (3) is true)

$\Rightarrow$ claim (5) is true $\forall x \in M$

$\Rightarrow$ Cor 1 is true $\#$

Ch5 Isometry, Space forms

(M, g) = complete Riemannian manifold (connected)

Def: (M, g) with constant sectional curvature is called a space form.

Thm 1: $\forall\, c \in \mathbb{R}$ & $n \geq 2$, $\exists$ unique (up to isometry) simply-connected space form of dimension n and with constant sectional curvature c.

egs (proof later)

- $C=0$ $\quad (\mathbb{R}^n,$ standard flat metric$)$

- $C=+1$ $\quad (S^n,$ standard metric$)$

- $C=-1$ $\quad \left(\mathbb{B}^n, \dfrac{4}{\left[1-\sum_{i=1}^{n}(x^i)^2\right]^2}(dx^1\otimes dx^1+\cdots+dx^n\otimes dx^n)\right)$

where $\mathbb{B}^n=\{(x^1,\dots,x^n): \sum_{i=1}^{n}(x^i)^2<1\}$

(Hyperbolic n-space : unit ball model)

Def: Let M be a submanifold of $\overline{M}$ equipped with the induced metric. Then M is called a <u>totally geodesic submanifold</u> of $\bar{M}$ if a geodesic γ (of $\bar{M}$) tangents to M implies $\gamma \subset M$.

<u>Note</u>: Such a geodesic γ of $\bar{M}$ must be a geodesic of the submanifold M.

<u>egs</u>:

- $\mathbb{R}^k \hookrightarrow \mathbb{R}^n : (x^1, \dots, x^k) \mapsto (x^1, \dots, x^k, 0, \dots, 0)$ gives a totally geodesic submanifold of $\mathbb{R}^n$.
- $S^n \subset \mathbb{R}^{n+1}$ is not a totally geodesic submanifold (since tangent lines to S^n don't stay in S^n.)

Let
- $M \subset \bar{M}$ be a submanifold
- M equipped with induced metric
- $D, \bar{D}$ = Levi-Civita connections of $M, \bar{M}$ respectively

(note : $D_X Y = (\bar{D}_X Y)^{\text{tangential part}}$)

Consider $S(X,Y) = D_X Y - \bar{D}_X Y$, $\forall X, Y \in \Gamma(TM)$

(note: S defined for vector fields on M, not $\bar{M}$)

<u>Facts</u>:
- $S(X_1 + X_2, Y) = S(X_1, Y) + S(X_2, Y)$
- $S(X, Y) = S(Y, X)$
- $\forall f \in C^\infty(M)$, $S(fX, Y) = S(X, fY) = f S(X, Y)$

The last one $\Rightarrow$ S defines a tensor field on M.

<u>Pf of Symmetry</u>: $S(X,Y) - S(Y,X)$

$$= (D_X Y - \bar{D}_X Y) - (D_Y X - \bar{D}_Y X)$$

$$= (D_X Y - D_Y X) - (\bar{D}_X Y - \bar{D}_Y X)$$

$$= [X,Y] - [X,Y] = 0 \quad (D, \bar{D} = \text{Levi-Civita})$$

others are easy (Ex.) . ※

Therefore, we can define a symmetric bilinear form on T_xM, $\forall x \in M$:

$$\forall v, w \in T_xM, \quad S(v,w) = S(V,W) \quad \left(S_x(v,w) = S(V,W)(x) \right)$$

where V, W = extension of v, w.

<u>Def</u>: This S is called the <u>2^{nd} fundamental form</u> of M in $\overline{M}$.

<u>Lemma 2</u>: $M \subset \overline{M}$ totally geodesic

$\Leftrightarrow S \equiv 0$, where $S = 2^{nd}$ f.f. of M in $\overline{M}$

(i.e. $D_X Y = \overline{D}_X Y$) for all $X, Y \in \Gamma(TM)$

<u>Pf</u>: ($\Rightarrow$) Let $x \in M$ & $v \in T_x M \subset T_x \overline{M}$

Let γ = geodesic on $\overline{M}$ with

$\gamma(0) = x$, $\gamma'(0) = v$.

$\Rightarrow \quad \bar{D}_{\gamma'}\gamma' = 0$

By assumption, γ is also a geodesic of M

$\Rightarrow \quad D_{\gamma'}\gamma' = 0$

Therefore $\quad S(v,v) = S(\gamma'(0), \gamma'(0))$

$$= D_{\gamma'}\gamma' - \bar{D}_{\gamma'}\gamma' = 0$$

Symmetry of $S \Rightarrow \quad S(v,w) = 0 \quad \forall\, v, w \in T_x M.$

($\Leftarrow$) Suppose $S \equiv 0$.

Let γ be a geodesic of $\bar{M}$ such that

$$\gamma(0) = x \text{ and } \gamma'(0) = v \in T_x M \subset T_x \bar{M}.$$

By Existence & Uniqueness of geodesic in M,

$\exists\ \xi =$ geodesic of M s.t.

$$\xi(0) = x\ ,\ \xi'(0) = v \in T_x M.$$

Then $S \equiv 0$

$$\Rightarrow\quad \bar{D}_{\xi'(t)}\xi'(t) = D_{\xi'(t)}\xi'(t) = 0 \quad (\xi = \text{geo. of } M)$$

$\Rightarrow$ ξ is also a geodesic of $\bar{M}$

Then uniqueness $\Rightarrow$ $\gamma = \xi \subset M$ #

Lemma 3 Let $M \subset \bar{M}$ be totally geodesic,

$K, \bar{K}$ = sectional curvatures of M, $\bar{M}$ respectively.

Then $\forall x \in M$, $\forall$ 2-plane $\Pi \subset T_xM \subset T_x\overline{M}$,

$$K(\Pi) = \overline{K}(\Pi)$$

(Pf: Immediately from Lemma 2)

<u>eg</u>: Let $\gamma : (a,b) \to \overline{M}$ be a smooth curve parametrized by arc-length. Suppose $\exists$ isometry $\varphi : \overline{M} \to \overline{M}$ s.t.

$$\gamma((a,b)) = \{ y \in \overline{M} : \varphi(y) = y \}$$

Then γ is a <u>normalized geodesic</u>.

Pf: We first note that $\forall$ geodesic ξ in $\overline{M}$,

$\varphi \circ \xi$ is also a geodesic in $\overline{M}$ (since φ = isom.)

Now $\forall t_0 \in (a,b)$, take a geodesic

$$\zeta \subset \bar{M} \quad \text{s.t.} \quad \begin{cases} \zeta(0) = \gamma(t_0) \\ \zeta'(0) = \gamma'(t_0) \end{cases}.$$

Since $\gamma((a,b)) =$ fixed point set of φ

$$d\varphi(\gamma'(t_0)) = \gamma'(t_0) \qquad (\text{by diff. } \varphi \circ \gamma = \gamma)$$

$$\Rightarrow \quad d\varphi(\zeta'(0)) = \zeta'(0)$$

$$\Rightarrow \quad (\varphi \circ \zeta)'(0) = \zeta'(0) \quad (\text{since } \varphi(\zeta(0)) = \zeta(0))$$

Uniqueness of geodesic $\Rightarrow \varphi \circ \zeta = \zeta$

$$\Rightarrow \quad \zeta \subset \{ y \in \bar{M} : \varphi(y) = y \} = \gamma((a,b))$$

$\Rightarrow \quad \gamma$ is <u>normalised</u> geodesic ※

<u>Lemma 4</u>: The set of fixed points of an isometry is a totally geodesic submanifold. (not necessary connected.)

<u>Pf</u>: Let $\varphi : \bar{M} \to \bar{M}$ be an isometry &

$M = \{ y \in \bar{M} : \varphi(y) = y \}$ be the set of fixed points of φ.

If M is submanifold of $\bar{M}$, then the same argument as in the previous example implies (Ex.)

M is totally geodesic. So we only need to show the following claim:

Claim: Let $x \in M$, $B(\delta) = \{ v \in T_x\overline{M} : |v| < \delta \}$

$$B_\delta = \{ y \in \overline{M} : d(x,y) < \delta \}$$

where $\delta > 0$ small enough s.t.

$$\exp_x : B(\delta) \longrightarrow B_\delta \text{ is a diffeomorphism}$$

$$\left(\Rightarrow \quad B_\delta = \exp_x B(\delta) \right)$$

Let $\mathcal{F} \subset T_x\overline{M}$ be a linear subspace defined by

$$\mathcal{F} = \{ v \in T_x\overline{M} : d\varphi(v) = v \}$$

Then

$$M \cap B_\delta = \exp_x \left(\mathcal{F} \cap B(\delta) \right).$$

Hence M is submanifold of $\overline{M}$.

Pf of Claim :

(1) $M \cap B_\delta \subset \exp_x(\mathcal{F} \cap B(\delta))$

Pf : Let $y \in M \cap B_\delta \subset B_\delta$

$\Rightarrow \exists\, v \in B(\delta)$ s.t. $\exp_x v = y$.

Let $\gamma(t) = \exp_x(tv) : [0,1] \to \overline{M}$

be the unique minimizing geodesic joining x to y.

Since $x, y \in M$, we have $\varphi(x) = x$ & $\varphi(y) = y$

$\Rightarrow \varphi \circ \gamma$ is also a minimizing geodesic joining x to y.

Uniqueness

$$\Longrightarrow \quad \varphi \circ \gamma = \gamma$$

$$\Longrightarrow \quad d\varphi(v) = v$$

$$\Longrightarrow \quad v \in \mathcal{J}$$

$$\therefore \quad y = \exp_x v \in \exp_x(\mathcal{J} \cap B(\delta))$$

This proves (1).

(2) $\exp_x(\mathcal{J} \cap B(\delta)) \subset M \cap B_\delta$

Pf: Let $y \in \exp_x(\mathcal{J} \cap B(\delta))$.

Then $\exists\ v \in \mathcal{J} \cap B(\delta)$ such that

$$y = \exp_x v.$$

Let $\gamma(t) = \exp_x(tv) : [0,1] \to \overline{M}$ be the unique minimizing geodesic joining x to y.

Since $v \in \mathcal{F}$, $d\varphi(\gamma'(0)) = \gamma'(0)$

$\Rightarrow$ $\varphi \circ \gamma$ & γ have the same initial values.

$\xrightarrow{\text{Uniqueness}}$ $\varphi \circ \gamma = \gamma$

$\Rightarrow$ $y = \gamma(1) = \varphi(\gamma(1)) = \varphi(y)$

$\Rightarrow$ $y \in M \cap B_\delta$.

$\#$

<u>Lemma 5</u>: $S^n \subset \mathbb{R}^{n+1}$ has constant sectional curvature $+1$, $\forall n \geq 2$.

<u>Pf</u>: "$n=2$" is proved in undergrad DG (ex)

If $n \geq 3$, define

$$\tilde{\varphi}: \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$$

$$(x^1, x^2, x^3, x^4 \dots x^{n+1}) \mapsto (x^1, x^2, x^3, -x^4, \dots, -x^{n+1})$$

Then $|\tilde{\varphi}(x)| = |x|$ (Euclidean norm)

Hence $\tilde{\varphi}$ induces an isometry

$$\varphi: S^n \to S^n.$$

The fixed points set

$$M = \{x \in S^n : \varphi(x) = x\}$$

$$= \{(x^1, x^2, x^3, 0, \dots, 0) : (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\}$$

$= S^2$

is a totally geodesic submanifold.

Hence $K_{S^n}(\Pi) = K_{S^2}(\Pi) = +1$,

$\forall$ 2-plane $\Pi \subset T_x S^2$, where $x \in (x^1, x^2, x^3, 0, \cdots, 0)$.

Repeat the argument for any 3 indices $i, j, k \in \{1, \cdots, n+1\}$

and the fact S^n is invariant under rotation,

we have proved that $K_{S^n} \equiv +1$. #

<u>Lemma 6</u> $\left(\mathbb{B}^n, \frac{4}{(1-|x|^2)^2} \sum_{i=1}^{n} dx^i \otimes dx^i\right)$ where $|x|^2 = \sum_{i=1}^{n} (x^i)^2$.

is a <u>complete</u> Riemannian manifold with constant sectional curvature -1.

<u>Pf</u>: (1) <u>Completeness</u>

<u>Pf</u>: First note that $\forall A \in O(n)$

$$A|_{\mathbb{B}^n} : \mathbb{B}^n \to \mathbb{B}^n \text{ is an isometry}$$

of the hyperbolic geometry

(A preserves $|x|$ & $\sum dx^i \otimes dx^i$)

Now consider the curve

$$\zeta(s) : (-\infty, \infty) \longrightarrow \mathbb{B}^n$$
$$\rotatebox{90}{$\in$}$$
$$s \longmapsto \left(\frac{e^s - 1}{e^s + 1}, 0, \cdots, 0 \right)$$

Then $\zeta'(s) = \left(\frac{2e^s}{(e^s+1)^2}, 0, \cdots, 0 \right)$

$$\Rightarrow \quad |\zeta'(s)|^2_{hyp} = \frac{4}{(1-|\zeta|^2)^2} \left[\frac{2e^s}{(e^s+1)^2} \right]^2 \underset{\text{(ex)}}{=} 1$$

$\therefore$ ζ is arc-length parametrized.

Let $A \in O(n)$ be given by

$$A(x^1, x^2, \cdots, x^n) = (x^1, -x^2, \cdots, -x^n).$$

Then $\quad \zeta((-\infty, \infty)) = \{ x \in \mathbb{B}^n : Ax = x \}$

Lemma 4 $\Rightarrow$ ξ is a normalized geodesic defined on the whole $(-\infty, \infty)$ with $\xi'(0)$ in the e_1-direction ($\{e_i\}$ = standard basis of $\mathbb{R}^n$)

Applying other $A \in O(n)$, we have geodesic with

$$(A\xi)'(0) = \text{any given direction}$$

defined on the whole $(-\infty, \infty)$

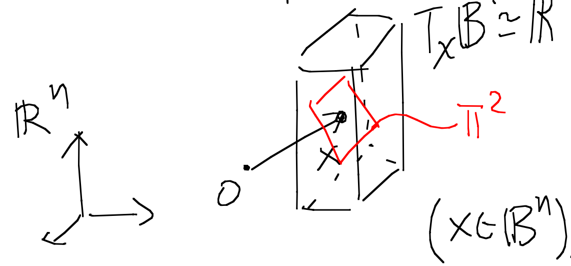
Therefore $\exp_0$ is defined on the whole $T_0 \mathbb{B}^n$.

Hence Hopf-Rinow $\Rightarrow$ $\mathbb{B}^n$ is complete. #

(2) <u>Curvature $\equiv -1$</u>

<u>Pf</u>: Let $x \in B^n$ and $\Pi \subset T_x B^n$ be a 2-plane.

Identify $T_x B^n \cong \mathbb{R}^n$,



and x can be considered as an element in $\mathbb{R}^n$ $(B^n \subset \mathbb{R}^n)$.

Assume $n \geq 3$.

Take a 3-dim'l subspace $E \subset \mathbb{R}^n$ s.t.

$$\mathrm{span}\{x, \Pi\} \subset E$$

(If $x \neq 0$ & $x \notin \Pi$, then E is unique, otherwise not)

Then $\mathbb{R}^n = E \oplus E^\perp$ orthogonal (in Euclidean)

and one can defines a map

$$\phi : (e, e') \mapsto (e, -e') \qquad e \in E, \ e' \in E^\perp$$

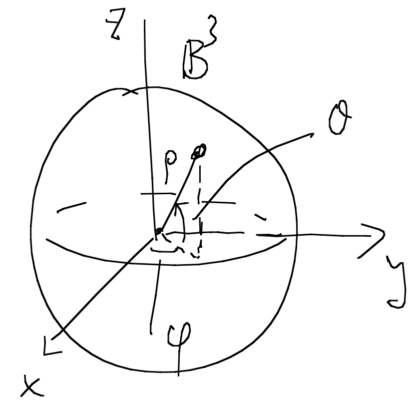
Then $\phi|_{\mathbb{B}^n}$ is an isometry of $\mathbb{B}^n$ with fixed point set $E \cap \mathbb{B}^n$.

$\Rightarrow$ $\mathbb{B}^3 = E \cap \mathbb{B}^n$ is a totally geodesic submanifold of $\mathbb{B}^n$

$$\Rightarrow \quad K_{\mathbb{B}^n}(\pi) = K_{\mathbb{B}^3}(\pi)$$

So we only need to show the case that $n=3$.

Let $\{\rho, \varphi, \theta\}$ = polar coordinates on $\mathbb{B}^3$



$\Rightarrow$ on $\mathbb{B}^3 \setminus \{0\}$, the metric

$$\frac{4}{(1-|x|^2)^2} \sum dx^i \otimes dx^i$$ can be

written as

$$\frac{4}{(1-\rho^2)^2} \left(d\rho^2 + \rho^2 d\theta^2 + \rho^2 \cos^2\theta \, d\varphi^2 \right)$$

where $d\rho^2 = d\rho \otimes d\rho$, ...

Let
$$\begin{cases} e_1 = \dfrac{1-\rho^2}{2}\dfrac{\partial}{\partial\rho} \\[2ex] e_2 = \dfrac{1-\rho^2}{2\rho}\dfrac{\partial}{\partial\theta} \\[2ex] e_3 = \dfrac{1-\rho^2}{2\rho\cos\theta}\dfrac{\partial}{\partial\varphi} \end{cases}$$

Then $\langle e_i, e_j\rangle = \delta_{ij}$ (Ex.)

$$\Rightarrow \quad \langle D_{e_i}e_j, e_k\rangle = \frac{1}{2}\Big\{\langle e_k, [e_i, e_j]\rangle + \langle e_j, [e_k, e_i]\rangle - \langle e_i, [e_j, e_k]\rangle\Big\}$$

Now
$$[e_1, e_2] = \frac{1-\rho^2}{2}\frac{\partial}{\partial\rho}\left(\frac{1-\rho^2}{2\rho}\frac{\partial}{\partial\theta}\right) - \frac{1-\rho^2}{2\rho}\frac{\partial}{\partial\theta}\left(\frac{1-\rho^2}{\rho}\frac{\partial}{\partial\rho}\right)$$
$$= \frac{1-\rho^2}{2}\left(\frac{1-\rho^2}{2\rho}\right)'\frac{\partial}{\partial\theta} = -\frac{1+\rho^2}{2\rho}e_2 \quad \text{(Ex)}$$

Similarly $[e_2, e_3] = \frac{1-\rho^2}{2\rho}\frac{\partial}{\partial\theta}\left(\frac{1-\rho^2}{2\rho\cos\theta}\frac{\partial}{\partial\varphi}\right) - \frac{1-\rho^2}{2\rho\cos\theta}\frac{\partial}{\partial\varphi}\left(\frac{1-\rho^2}{2\rho}\frac{\partial}{\partial\theta}\right)$

$$\overset{(ex)}{=\!=} \frac{1-\rho^2}{2\rho}\tan\theta\, e_3$$

$$[e_1, e_3] = -\frac{1+\rho^2}{2\rho} e_3 \qquad (Ex.)$$

In conclusion

$$\begin{cases} [e_1, e_2] = -\frac{1+\rho^2}{2\rho} e_2 \\ [e_2, e_3] = \frac{1-\rho^2}{2\rho}\tan\theta\, e_3 \\ [e_1, e_3] = -\frac{1+\rho^2}{2\rho} e_3 \end{cases}$$

Then straight faward calculation (Ex) $\Longrightarrow$

$$
\begin{cases}
D_{e_1} e_1 = 0, & D_{e_2} e_1 = \dfrac{1+\rho^2}{2\rho} e_2, & D_{e_3} e_1 = \dfrac{1+\rho^2}{2\rho} e_3 \\[2ex]
D_{e_1} e_2 = 0, & D_{e_2} e_2 = -\dfrac{1+\rho^2}{2\rho} e_1, & D_{e_3} e_2 = -\dfrac{1-\rho^2}{2\rho} \tan\theta \, e_3 \\[2ex]
D_{e_1} e_3 = 0, & D_{e_2} e_3 = 0, & D_{e_3} e_3 = -\dfrac{1+\rho^2}{2\rho} e_1 + \dfrac{1-\rho^2}{2\rho} \tan\theta \, e_2
\end{cases}
$$

Hence

$$
\begin{aligned}
R(e_1, e_2, e_1, e_2) &= \langle R_{e_1 e_2} e_1, e_2 \rangle \\
&= \langle D_{[e_1, e_2]} e_1 - [D_{e_1}, D_{e_2}] e_1, e_2 \rangle \\
&= -\frac{1+\rho^2}{2\rho} \langle D_{e_2} e_1, e_2 \rangle - \langle D_{e_1}(D_{e_2} e_1) - D_{e_2}(\cancel{D_{e_1} e_2}), e_2 \rangle
\end{aligned}
$$

$$= -\left(\frac{1+\rho^2}{2\rho}\right)^2 \langle e_2, e_2\rangle - \left\langle D_{e_1}\left(\frac{1+\rho^2}{2\rho} e_2\right), e_2 \right\rangle$$

$$= -\left(\frac{1+\rho^2}{2\rho}\right)^2 - e_1\left(\frac{1+\rho^2}{2\rho}\right)\langle e_2, e_2\rangle - \frac{1+\rho^2}{2\rho}\langle \cancel{D_{e_1}} e_2, e_2\rangle$$

$$= -\left(\frac{1+\rho^2}{2\rho}\right)^2 - \frac{1-\rho^2}{2}\frac{\partial}{\partial\rho}\left(\frac{1+\rho^2}{2\rho}\right)$$

$$= -1 \quad (\text{Ex.})$$

Similarly $R(e_1, e_3, e_1, e_3) = R(e_2, e_3, e_2, e_3) = -1$ (Ex.)

To complete the proof, we need to show that all other

$$R(e_i, e_j, e_k, e_l) = 0. \quad (\text{Ex})$$

Since $n=3$, the indices have to be repeated.

It is clear that if $i=j=k=l$ or 3 of the indices are equal, then $R(e_i, e_j, e_k, e_l) = 0$.

Therefore, we only need to consider

$$R(e_i, e_j, e_i, e_k) \quad \text{with } j<k. \ (i,j,k \text{ distinct})$$

Other cases are clear zero or can be reduced to this case. (If $j=k$, it is the previous situation)

For $i=3$,

$$R(e_3, e_1, e_3, e_2) = \langle R_{e_3 e_1} e_3, e_2 \rangle$$

$$= \langle D_{[e_3,e_1]} e_3, e_2 \rangle - \langle D_{e_3} D_{e_1} \cancel{e_3}, e_2 \rangle + \langle D_{e_1} D_{e_3} e_3, e_2 \rangle$$

$$= \frac{1+\rho^2}{2\rho} \langle D_{e_3} e_3, e_2 \rangle + \langle D_{e_1}(D_{e_3} e_3), e_2 \rangle$$

$$= \frac{1+\rho^2}{2\rho} \left\langle -\frac{1+\rho^2}{2\rho} e_1 + \frac{1-\rho^2}{2\rho} \tan\theta \, e_2 , e_2 \right\rangle$$

$$+ \left\langle D_{e_1}\left(-\frac{1+\rho^2}{2\rho} e_1 + \frac{1-\rho^2}{2\rho} \tan\theta \, e_2 \right), e_2 \right\rangle$$

$$= \frac{(1+\rho^2)(1-\rho^2)}{4\rho^2} \tan\theta + e_1\left(\frac{1-\rho^2}{2\rho} \tan\theta \right)$$

$$= \frac{1-\rho^4}{4\rho^2} \tan\theta + \frac{1-\rho^2}{2} \left(\frac{1-\rho^2}{2\rho} \right)' \tan\theta$$

$$= 0 \qquad (\text{Ex})$$

Similarly, $R(e_1, e_2, e_1, e_3) = R(e_2, e_1, e_2, e_3) = 0$.

Hence $\mathbb{B}^3$ has sectional curvature $\equiv -1$.

This proves Lemma 6. ※

<u>Existence of Thm 1</u> : By Lemmas 5 & 6, we have complete simply-connected Riemannian manifolds of any dimension ≥ 2 with constant sectional curvature $= \pm 1$. By scaling, we have

$$K_{\cancel{\frac{1}{\sqrt{c}}}\,\frac{1}{c}\, g} = c K_g \qquad (\forall \text{ metric } g \quad (\text{Ex}))$$

$$= \pm c$$

Together with $\mathbb{R}^n$, we've proved the existence part of Thm 1.

#

§5.2 Geodesic & curvatures

Let $\mathbb{H}^n = \left(\mathbb{B}^n, \frac{4}{(1-|x|^2)^2} \sum_{i=1}^n dx^i \otimes dx^i\right)$.

<u>Facts</u>: $\mathbb{R}^2 \hookrightarrow \mathbb{R}^n$, $S^2 \hookrightarrow S^n$, $\mathbb{H}^2 \hookrightarrow \mathbb{H}^n$

are totally geodesic submanifolds, the studies of geodesics on $\mathbb{R}^n$, S^n & $\mathbb{H}^n$ can be reduced to $\mathbb{R}^2$, S^2, & $\mathbb{H}^2$ (since for any $x, y \in \mathbb{R}^n, S^n$ or $\mathbb{H}^n$, $\exists$ isometry of $\mathbb{R}^n$, S^n or $\mathbb{H}^n$ respectively, taking x to y. (Ex))

Let $M = \mathbb{R}^2, S^2,$ or $\mathbb{H}^2$ & $0 \in M$ be a fixed point.

Let $C(r) = \{x \in M : d(0,x) = r\}$ be the geodesic circle of radius r.

If $r>0$ small enough, then

$$C(r) = \exp_0 \left(\text{circle of radius } r \text{ in } T_0M \right)$$

Denote:

$$\text{length } C(r) = \begin{cases} C_0(r), & \text{if } M = \mathbb{R}^2 \\ C_+(r), & \text{if } M = S^2 \\ C_-(r), & \text{if } M = \mathbb{H}^2 \end{cases}$$

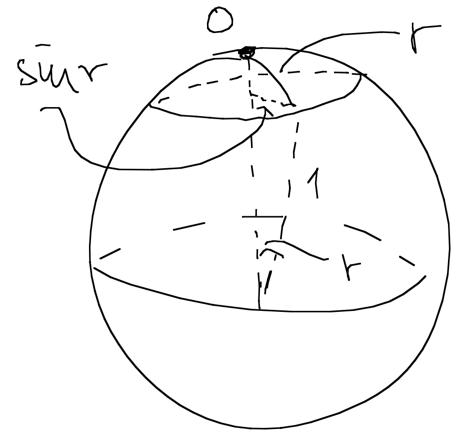
If $M = \mathbb{R}^2$, it is clear that

$$\boxed{C_0(r) = 2\pi r}$$

If $M = S^2$, we may assume 0 = north pole.

Then geodesic circle

$C(r)$ = a circle of radius $\sin r$ in $\mathbb{R}^3$



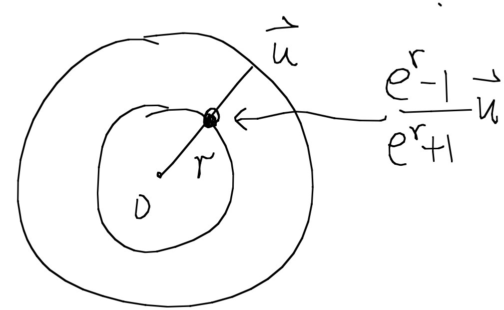
$\Rightarrow$ $\boxed{C_+(r) = 2\pi \sin r}$

If $M = \mathbb{H}^2$, then by the proof of lemma 6,

a normal geodesic from O is given by

$$\gamma(s) = \frac{e^s - 1}{e^s + 1} \vec{u}, \quad \text{where } \vec{u} = \text{unit vector in } \mathbb{R}^2.$$

$$\left(|\gamma'(s)|_{\mathbb{H}^2} = 1 \right)$$



$$\Rightarrow \quad d_{\mathbb{H}^2}(0, \gamma(r)) = \int_0^r |\gamma'(s)|_{\mathbb{H}^2}\, ds = r$$

$\Rightarrow$ $C(r)$ = Euclidean circle of radius $\frac{e^r - 1}{e^r + 1}$ $\left(= \tanh \frac{r}{2} \right)$

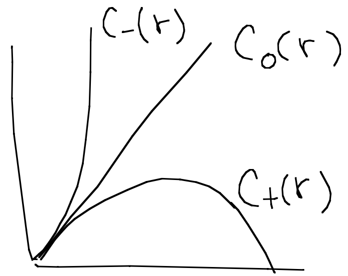
$$\Rightarrow \quad C_-(r) = \int_0^{2\pi} \frac{2}{1-\rho^2} \cdot \rho\, d\theta \quad \text{where } \rho = \tanh \frac{r}{2}$$

$$= 2\pi \cdot \frac{2\rho}{1-\rho^2}$$

$$\Rightarrow \quad \boxed{C_-(r) = 2\pi \sinh r}$$

In summary, we have

$$\begin{cases} C_0(r) = 2\pi r \\ C_+(r) = 2\pi \sin r \\ C_-(r) = 2\pi \sinh r \end{cases}$$



To generalize the above to arbitrary complete Riem. manifold, we need to study variations of geodesic.

Let $\gamma : [a,b] \times [c,d] \to M$ be a C^∞ map from the rectangle $[a,b] \times [c,d]$ to a complete Riem manifold M (of any dimension ≥ 2). Denote a point in $[a,b] \times [c,d]$ by (t, u). Then we can define 2 tangent vector fields along γ by