<u>Note</u>: $\exp_x : B(\omega) \subset T_xM \longrightarrow M$ with $\exp_x(0) = x$.

Therefore $(d\exp_x)_0 : T_0(T_xM) \to T_xM$

Since T_xM is linear,

$$T_0(T_xM) \cong T_xM$$

(In fact, $\forall\, v \in T_xM$, we define

$\xi_v : t \mapsto tv$ a curve in T_xM

with $\xi_v(0) = 0$ & "$\xi_v'(0) = v$")

Hence $(d\exp_x)_0$ can be regarded as a map from T_xM to itself.

Pf of lemma : $\forall\ v \in T_xM \cong T_0(T_xM)$

$$(d\exp_x)_0(v) = \frac{d}{dt}\Big|_{t=0} \exp_x(tv) \qquad \text{(identification of } T_0(T_xM) \cong T_xM\text{)}$$

$$= \frac{d}{dt}\Big|_{t=0} \gamma_{tv}(1) \qquad \text{(definition of } \exp_x\text{)}$$

$$= \frac{d}{dt}\Big|_{t=0} \gamma_v(t) \qquad \text{(ex.)}$$

$$= \gamma_v'(0) = v$$

※

We can even have a stronger result :

Thm: $\forall$ compact $K \subset M$, $\exists\, \varepsilon > 0$ s.t.

$\forall x \in K$, $\exp_x$ is diffeo on $B_x(\varepsilon)$.

(This shows that we can find a uniform ε $\forall$ cpt. $K \subset M$)

Pf: It is sufficient to show that

$$\boxed{\begin{array}{l} \forall x \in M, \exists\, \varepsilon > 0, \text{ \& open nhd. } \Omega \text{ of } x \text{ s.t.} \\ \forall y \in \Omega,\ \exp_y \text{ is a diffeo. on } B_y(\varepsilon) \subset T_yM, \end{array}}$$

By Thm(#), $\exists$ nhd $\mathcal{U}$ of x s.t.

$\exp_y$ is defined on some ball $B_y(\varepsilon(y))$, $\varepsilon(y) > 0$.

Let $N = \{(y, v) : y \in \mathcal{U}, v \in B_y(\varepsilon(y))\} \subset TM$, and define

$$\begin{array}{cccc} E: & N & \longrightarrow & M \times M \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & (y,v) & \longmapsto & (y, \exp_y v) \end{array}$$

By the theory of ODE, E is C^∞.

Choose a coordinate system $\{x^1, \dots, x^n\}$ centered at x (i.e. $x^i(x) = 0$). Then any (y,v) can be represented by coordinates $(x^1, \dots, x^n, u^1, \dots, u^n)$ where $\{u^i\}$ are given by $v = \sum u^i \frac{\partial}{\partial x^i}$.

(i.e. $u^i = dx^i(v)$, $\forall i$)

$\Rightarrow \left\{ \frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial u^1}, \cdots, \frac{\partial}{\partial u^n} \right\}$ is a basis of the tangent space $T_{(y,v)}(TM)$ of TM.

Now

$$dE_{(x,0)}\left(\frac{\partial}{\partial x^i}\Big|_{(x,0)} \right) = \frac{d}{dt}\Big|_{t=0} E(\xi_i(t), 0)$$

where $\xi_i(t)$ is a curve in M s.t.

$$\xi_i(0) = x \quad \& \quad \xi_i'(0) = \frac{\partial}{\partial x^i}$$

(i.e. $t \mapsto (\xi_i(t), 0)$ curve in TM)

$$\Rightarrow \quad dE_{(x,0)}\left(\frac{\partial}{\partial x^i}\Big|_{(x,0)} \right) = \frac{d}{dt}\Big|_{t=0} \left(\xi_i(t), \exp_{\xi_i(t)}(0) \right)$$

$$= \frac{d}{dt}\Big|_{t=0} \left(\xi_i(t), \xi_i(t) \right)$$

$$= \left(\frac{\partial}{\partial x^i}\Big|_x , \frac{\partial}{\partial x^i}\Big|_x \right)$$

Also $dE_{(x,0)}\left(\frac{\partial}{\partial u^i}\Big|_{(x,0)} \right) = \frac{d}{dt}\Big|_{t=0} E\left(x, t \frac{\partial}{\partial x^i}\Big|_x \right)$

$$= \frac{d}{dt}\Big|_{t=0} \left(x, \exp_x\left(t \frac{\partial}{\partial x^i} \right) \right)$$

$$= \left(0, (d\exp_x)_0 \left(\frac{\partial}{\partial x^i} \right) \right)$$

$$= \left(0, \frac{\partial}{\partial x^i}\Big|_x \right) \quad \text{by previous lemma.}$$

$\Rightarrow \quad dE_{(x,0)} : T_{(x,0)} N \to T_x M \times T_x M$ is nonsingular.

$\therefore$ IFT $\Rightarrow$ E is a local diffeo. that maps a nbd $\mathcal{W}$ of $(x,0)$ in TM to a nbd of

$$(x, \exp_x(0)) = (x,x) \text{ in } M \times M.$$

Therefore, $\exists\, c > 0$, $\varepsilon' > 0$ s.t.

$$\{ (y,v) \in TM : |x^i(y)| \le c,\ |u^i(v)| \le \varepsilon' \}$$

is a cpt. subset of $\mathcal{W}$.

norm wrt metric g.

$\Rightarrow \exists\, \varepsilon > 0$ s.t.

$$\{ (y,v) \in TM : |x^i(y)| \le c,\ |v| \le \varepsilon \} \subset \mathcal{W}$$

Then this $\varepsilon>0$, & $\Omega=\{y\in\mathcal{U}: |x^i(y)|\leq c\}$ satisfy the requirement. #

4.2 Gauss Lemma, minimizing geodesic.

Let (M,g) be a Riemannian manifold and $x\in M$ be fixed. Let $\delta>0$ sufficiently small such that $\exp_x$ is a diffeomorphism on $B(\delta)=\{v\in T_xM: |v|<\delta\}$, where $|v|=\langle v,v\rangle^{1/2}$. Denote

$$B_\delta=\exp_x(B(\delta))$$

Then • $\gamma(t) = \exp_x(tv)$, $t \in [0,1]$, $v \in B(\delta)$

is called a radial geodesic (segment)

joining x to $\exp_x(v)$.

And $\forall\, t \in (0, \delta)$,

- $S_t = \exp_x\left(\{ v \in T_xM : |v| = t \} \right)$ is called the geodesic sphere of radius t centered at x.

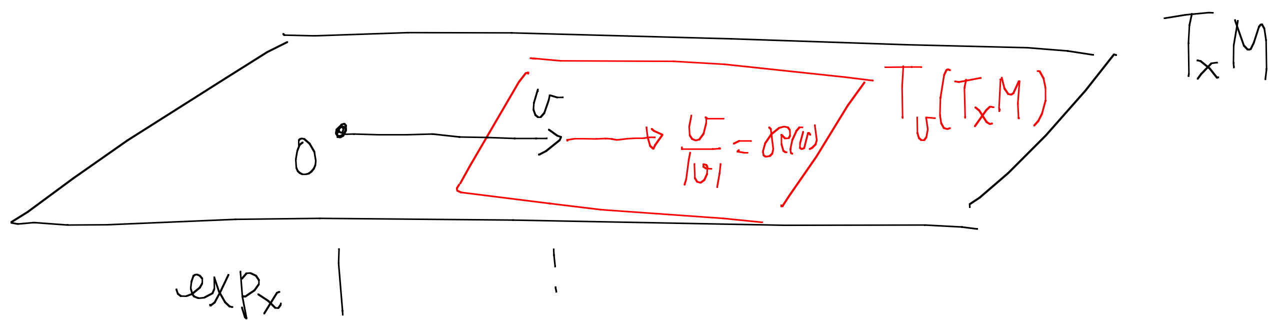
- $B_t = \exp_x(B(t))$ is called the geodesic ball of radius t centered at x.

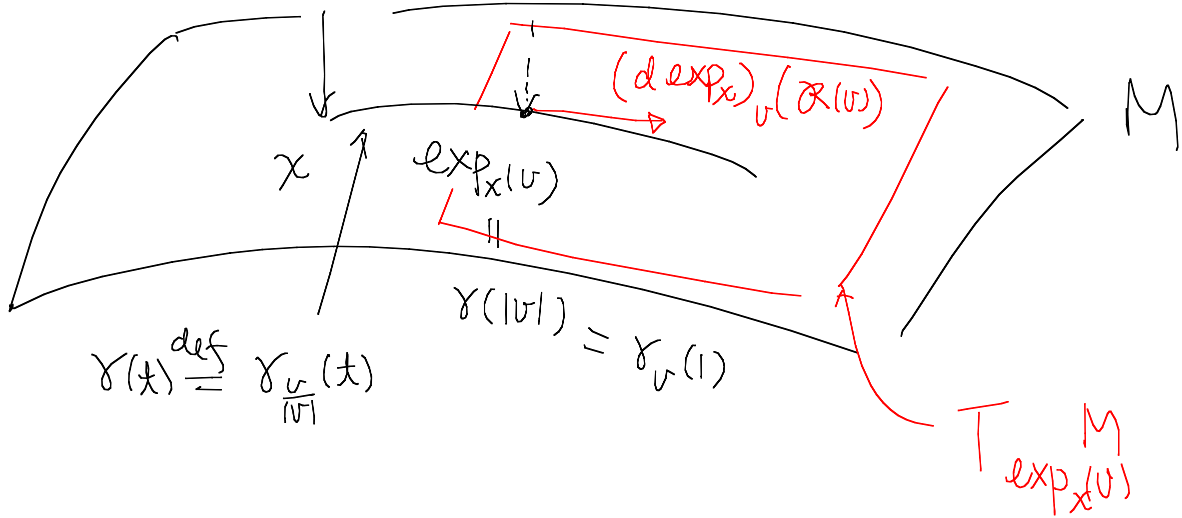
Lemma: (M, g), x, δ as above. Define a vector field $\mathcal{R}$ on $T_xM \setminus \{0\}$ by

$$\mathcal{R}(v) = \frac{v}{|v|} \quad \left(\mathcal{R} : T_xM \setminus \{0\} \to T(T_xM \setminus \{0\}) \right)$$

with $T_v(T_xM \setminus \{0\}) \cong T_xM$

Then $\left| (d \exp_x)_v (\mathcal{R}(v)) \right| = 1$.





Pf: For $v \in T_xM \setminus \{0\}$, let $\gamma(t) = \gamma_{\frac{v}{|v|}}(t)$ the normalized geodesic on M s.t. $\gamma(0) = x$, $\gamma'(0) = \frac{v}{|v|}$

By definition of $\exp_x$,

$$\exp_x(v) = \gamma(|v|)$$

Since $\mathcal{R}(v) =$ unit tangent vector of the line

$v + t\,\mathcal{R}(v)$.

$$(d\exp_x)_v(\mathcal{R}(v)) = \frac{d}{dt}\Big|_{t=0}(\exp_x)(v + t\,\mathcal{R}(v))$$

$$= \frac{d}{dt}\Big|_{t=0}(\exp_x)\left((|v|+t)\frac{v}{|v|}\right)$$

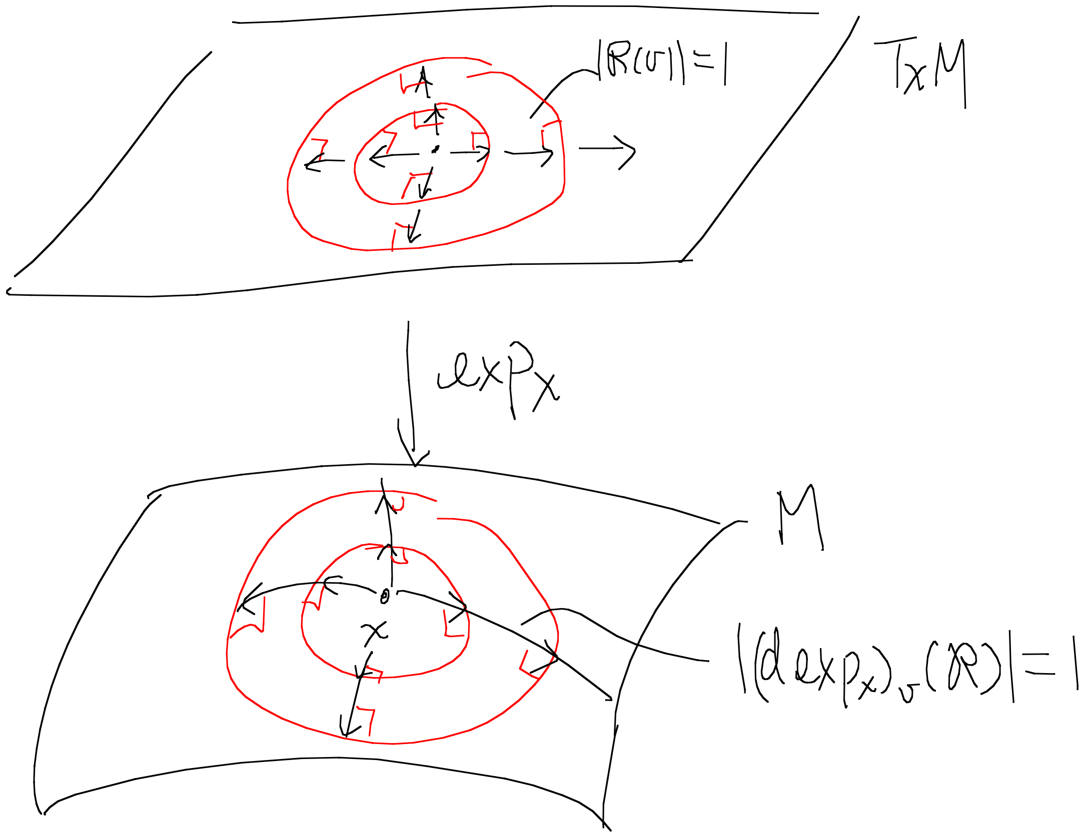
$$= \frac{d}{dt}\Big|_{t=0}\gamma(|v|+t)$$

$$= \gamma'(|v|)$$

$$\therefore \quad |(d\exp_x)_v(\mathcal{R}(v))| = |\gamma'(|v|)| = |\gamma'(0)| = 1 \quad \#$$

<u>Gauss Lemma</u> : Radial geodesic are <u>athogonal</u> to

the geodesic sphere S_t, $\forall t \in (0, \delta)$.



Pf: Define a diffeo

$$F : S^{n-1} \times (0, \delta) \xrightarrow{\subset T_x M} B_\delta \setminus \{x\}$$

$$\begin{array}{ccc} \cup & & \\ (p, t) & \longmapsto & F(p,t) = \exp_x(tp) \end{array}$$

Then for fixed $t \in (0, \delta)$

$$F(\cdot, t) : S^{n-1} \times \{t\} \to S_t$$

is a diffeomorphism.

Let γ = radial geodesic intersecting S_t at the point $\exp_x(tp)$.

We take a local coordinate $\{y^1, \dots, y^{n-1}\}$ around $p \in S^{n-1}$. And let r be the natural parameter of the interval $(0, \delta)$.

Then $\begin{cases} R = dF\left(\frac{\partial}{\partial r}\right) \\ Y_i = dF\left(\frac{\partial}{\partial y^i}\right) \end{cases}$

are vector fields on $B_\delta \setminus \{x\} \subset M$ s.t.

Y_i are tangential to S_t (and form a basis of $T_y S_t$ (for $y \in S_t \subset B_\delta \setminus \{x\}$))

and R is tangential to a radial geodesic.

Therefore, we need to show that $\langle R, Y_i \rangle = 0 \ \forall i$

at $\exp_x(tp)$.

Consider $\langle R, Y_i \rangle$ along the radial geodesic γ.

derivative wrt parameter of γ (i.e. $r \in (0, \delta)$)

Then

$$\langle R, Y_i \rangle'$$

$$= R \langle R, Y_i \rangle$$

$$= \langle D_R R, Y_i \rangle + \langle R, D_R Y_i \rangle$$

$$= 0 + \langle R, D_{Y_i} R \rangle + \langle R, [R, Y_i] \rangle$$

$$\left(\text{Since } D_R R = D_{\gamma'} \gamma' = 0 \right)$$

However

$$[R, Y_i] = \left[dF\left(\frac{\partial}{\partial r}\right), dF\left(\frac{\partial}{\partial y_i}\right) \right]$$

(ex.)

$$= dF\left(\left[\frac{\partial}{\partial r}, \frac{\partial}{\partial y_i} \right] \right)$$

$$= 0$$

Hence $\langle R, Y_i \rangle' = \langle R, D_{Y_i} R \rangle = \frac{1}{2} Y_i \langle R, R \rangle$

$= 0$ (by lemma that $|R| = 1$)

$\Rightarrow \langle R, Y_i \rangle = \lim_{r \to 0} \langle R, Y_i \rangle (\gamma(r))$

$= 0$ since $|Y_i| \to 0$ as $\gamma(r) \to x$

($S_t \to \{x\}$ as $t \to 0$)

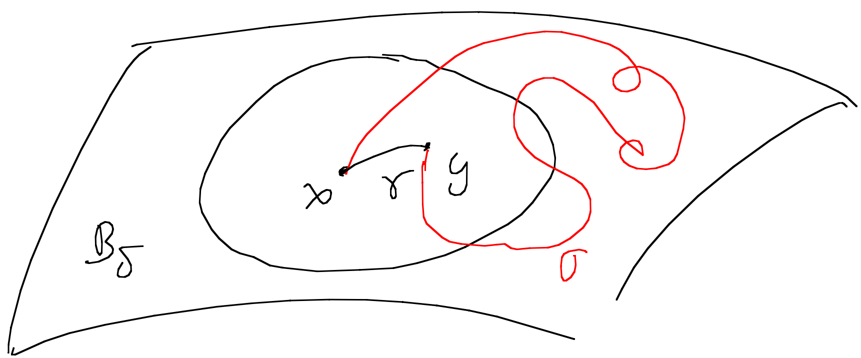
$\#$

<u>Thm</u>: Let
- (M, g) = Riemannian manifold
- $x \in M$
- $\delta > 0$ s.t. $\exp_x : B(\delta) \to B_\delta$ is a diffeo.

- γ = unique radial geodesic joining x and a point $y \in B_\delta \setminus \{x\}$

Then $L(\gamma) \leq L(\sigma)$ for all piecewise smooth curve σ on M (not necessary within B_δ) joining x to y

Equality holds $\Leftrightarrow$ σ = monotonic reparametrization of γ.

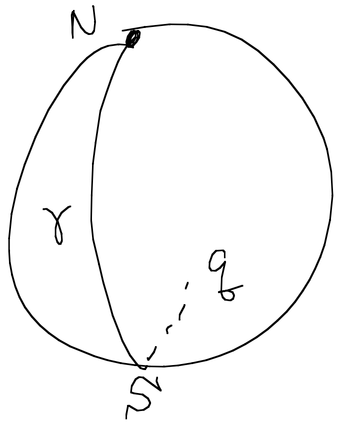


Cor: Let $\gamma : [0, c] \to M$ be a arc-length parametrized piecewise smooth curve such that $L(\gamma) \leq L(\sigma)$ $\forall$ piecewise smooth curve σ joining $\gamma(0)$ & $\gamma(c)$.

Then γ is a geodesic.

Caution: The converse of the Cor. is not true in general.

eg



γ = geodesic, but not length minimizing.

Def: A geodesic $\gamma = [0, c] \to M$ is called a <u>minimizing geodesic</u> if $L(\gamma) \le L(\sigma)$ $\forall \sigma$ joining $\gamma(0)$ & $\gamma(c)$.

<u>Pf of Cor</u> (<u>Assuming the Thm</u>)

Let $x = \gamma(0)$. Choose B_δ as in thm.

Let $t_1 = \min\{t : \gamma(t) \in \partial B_\delta\}$. (If t_1 doesn't exist, then we are done.)

If $\gamma|_{[0,t_1]}$ is not geodesic, then by the thm,

$L(\gamma|_{[0,t_1]}) > L(\gamma_1)$

where γ_1 = radial geodesic joining $x = \gamma(0)$ & $\gamma(t_1)$ in B_δ.

$\Rightarrow L(\gamma_1 \cup \gamma|_{[t_1, c]}) < L(\gamma)$

which is a contradiction.

Hence $\gamma|_{[0,t_1]}$ is a geodesic.

Continuing this argument $\Rightarrow \gamma|_{[0,c]}$ is a geodesic.

\#\#

Pf: (of ~~Gauss Lemma~~ of the Thm)

As in the proof of the Gauss Lemma, we can find basis $\{R, Y_1, \dots, Y_{n-1}\}$ of T_zM for $z \in B_\delta \setminus \{x\}$

s.t. R = tangential to the radial direction & in fact $|R| = 1$

$Y_1, \dots, Y_{n-1}$ = tangential to the geodesic sphere.

WLOG, we may assume $\sigma \subset B_\delta$.

Then for any such $\sigma : [0,1] \to B_\delta$ s.t.

$$\sigma(0) = x, \quad \sigma(1) = y,$$

we have $\forall t \in [0,1]$

$$\sigma'(t) = f(t) R(\sigma(t)) + T(t)$$

for some function $f(t)$, where

$$T(t) = \text{a linear combination of } Y_i\text{'s.}$$

Let $v \in B(\delta)$ be the unique vector s.t.

$$\exp_x(v) = y$$

Then $\quad \xi = \exp_x^{-1} \circ \sigma \quad$ is a curve in $B(\delta) \subset T_x M$

joining 0 and v.

Since $(d\exp_x^{-1})(R) = \mathcal{R}$ $\Big($= unit radial vector field defined above.$\Big)$

$(d\exp_x^{-1})(Y_i)$ tangential to $S^{n-1}_{|\xi(t)|} \subset T_x M$, we see that (by Gauss lemma)

$$(d\exp_x^{-1})(\langle \sigma', R\rangle R) = f R$$

is the radial projection of the tangent vector ξ'.

$$\Rightarrow \quad |v| = |\xi(1)| - |\xi(0)| = \int_0^1 f(t)\,dt$$

$$\Rightarrow \quad L(\gamma) = \int_0^1 f(t)\,dt \quad \text{(since } \gamma \text{ is the radial geodesic)}$$

Gauss lemma $\underbrace{}_{\text{again}} \Rightarrow$

$$|\sigma'(t)|^2 = f^2(t)\,|R(\sigma(t))|^2 + |T(t)|^2$$

$$= f^2(t) + |T(t)|^2$$

$$\Rightarrow \quad L(\sigma) = \int_0^1 |\sigma'|$$

$$= \int_0^1 \sqrt{f^2(t) + |T(t)|^2}\, dt$$

$$\geq \int_0^1 f(t)\, dt = L(\gamma).$$

Finally, if $L(\sigma) = L(\gamma)$, then $T(t) = 0$ & $f > 0$.

$$\Rightarrow \quad \sigma'(t) = f(t)\, R(\sigma(t)) \quad \text{with } f > 0$$

$\Rightarrow \quad \sigma$ = monotonic reparametrization of γ. ※

4.3 Completeness, metric structure.

(M, g) = Riemannian manifold (connected)

Def: $d : M \times M \to [0, \infty)$ defined by

$$d(x,y) = \inf_{\gamma} L(\gamma),$$

where "inf" is taken over all piecewise smooth curves γ joining x and y, is called the distance (a metric) of (M, g).

Thm: (M, d) is a metric space, i.e. d satisfies

(1) $d(x,y) \geq 0$; "$=$" iff $x=y$.

(2) $d(x,y) = d(y,x)$,

(3) $d(x,y) \leq d(x,z) + d(z,y)$.

Pf: All are easy (Exs.) and we prove only "$d(x,y)=0 \Rightarrow x=y$".

Suppose $x \neq y$. If $y \in B_\delta$, where δ is given as in the "Thm" in the previous section, then

$d(x,y) = L(\gamma)$, where γ = radial geodesic from x to y.

$\Rightarrow d(x,y) > 0$,

Continuity argument $\Rightarrow d(x,y) = \delta > 0$ if $y \in \partial B_\delta$.

Hence if $y \notin B_\delta$, and σ = curve joining x to y,

Choose the 1^{st} point y_1 of σ on ∂B_σ and conclude that

$$L(\sigma) \geq L\left(\sigma\Big|_{(\text{from } x \text{ to } y_1)}\right) \geq \delta > 0$$

Taking "inf" $\Rightarrow d(x,y) \geq \delta > 0$ ※

In fact, we have a stronger theorem

<u>Thm</u>: The topology of (M, d) is the same as the

original topology of M.

(Pf: Ex a pages 61-62 of H. Wu a do Carmo.)

Def: A Riemmann manifold (M, g) is said to be complete if the associated metric space (M, d) is complete.

egs: $(\mathbb{R}^n$, standard metric$)$, $(S^n$, standard metric$)$ are complete

Hopf-Rinow Thm: The following statements are equivalent on a Riemannian manifold (M, g):

(1) M is complete;

(2) $\forall\ x \in M$, $\exp_x$ defined on the whole T_xM;

(3) $\exists\ x \in M$, $\exp_x$ defined on the whole T_xM;

(4) bounded closed subsets of M are compact.

<u>Cor 1</u> of <u>Hopf-Rinow Thm</u>

If (M, g) is complete, then $\forall\ x \neq y \in M$,

$\exists$ a minimizing geodesic γ joining x and y.

(Recall: all manifolds discussed in this course are assumed to be connected.)

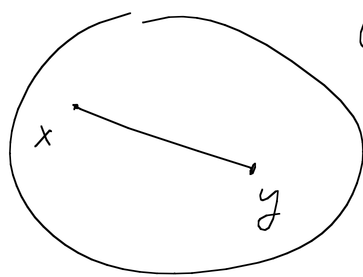
Cor 2 : If (M, g) is complete, then $\forall x \in M$,

$$\exp_x : T_x M \to M \quad \text{is surjective.}$$

Notes :

- The converse of Cor 1 of Hopf-Rinow Thm is not true in general :

eg



convex in $\mathbb{R}^n$

$\uparrow$

incomplete

- A general complete metric space may not have Heine-Borel property ((4) of the thm) eg: $S = \{a_1, a_2, \dots\}$ countable infinite set of distinct elements.

Define discrete metric d on S by

$$d(a_i, a_j) = 1 - \delta_{ij}$$

Then (S, d) is a complete metric space which is bounded.

$\Rightarrow$ S is a closed & bounded set but not compact.

Pf of Hopf-Rinow Thm:

(1) $\Rightarrow$ (2) Let $\gamma : [0, \delta) \to M$ be a geodesic