Furthermore, if $D$ is the Levi-Civita connection of a metric $g$ on $M$, then $\forall$ 2 parallel vector fields $X$ & $Y$ along $\gamma$ ($\gamma$ embedded)

$$\begin{aligned}
\frac{d}{dt}\langle X, Y\rangle &= \gamma'(t)\langle X, Y\rangle \\
&= \langle \cancelto{0}{D_{\gamma'(t)} X}, Y\rangle + \langle X, \cancelto{0}{D_{\gamma'(t)} Y}\rangle \\
&= 0
\end{aligned}$$

$\therefore$ $P^{\gamma} : T_{\gamma(a)}M \to T_{\gamma(b)}M$ is in fact an <u>isometry</u> of the inner product spaces.

Conversely, if $D$ is a connection such that all $P^{\gamma}$ are isometries of the inner product spaces, then $\forall$ vector fields $X, Y, Z$, we choose a curve $\gamma : [0,1] \to M$

s.t. $\gamma'(0) = X(x)$ $(x \in M)$

Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of $T_xM$. Then parallel transport $P^\gamma$ along $\gamma$ defines orthonormal basis $\{e_1(t), \cdots, e_n(t)\}$ of $T_{\gamma(t)}M$, $\forall t \in [0,1]$ (since $P^\gamma$ are isometries $\forall t$)

Hence
$$Y(\gamma(t)) = \sum Y^i(t)\, e_i(t)$$
$$Z(\gamma(t)) = \sum Z^i(t)\, e_i(t)$$
for some $Y^i(t)$ & $Z^i(t)$

$$\Rightarrow \quad X(x)\langle Y, Z\rangle = \gamma'(0)\langle Y, Z\rangle$$
$$= \left.\frac{d}{dt}\right|_{t=0} \langle Y, Z\rangle(\gamma(t))$$

$$= \frac{d}{dt}\Big|_{t=0} Y^i(t) Z^i(t)$$

$$= \frac{dY^i}{dt}(0)\, Z^i(0) + Y^i(0) \frac{dZ^i}{dt}(0)$$

Note that $D_{\gamma'(0)} Y = D_{\gamma'(0)} \left( \sum Y^i(t) e_i(t) \right)$

$$= \sum \frac{dY^i}{dt}(0)\, e_i + \sum Y^i(0) \cancelto{0}{D_{\gamma'(0)} e_i}$$

$$= \sum \frac{dY^i}{dt}(0)\, e_i$$

Similarly for $D_{\gamma'(0)} Z$.

$\Rightarrow \quad X\langle Y, Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle.$

$\Rightarrow \quad D$ is compatible with the metric $g$.

<u>Conclusion</u> : $D$ = compatible with $g \iff P^\gamma = \text{isometry}, \ \forall \gamma$.

In particular, if $D$ is symmetric,

$$D = \text{Levi-Civita} \iff P^\gamma = \text{isometry}, \forall \gamma.$$

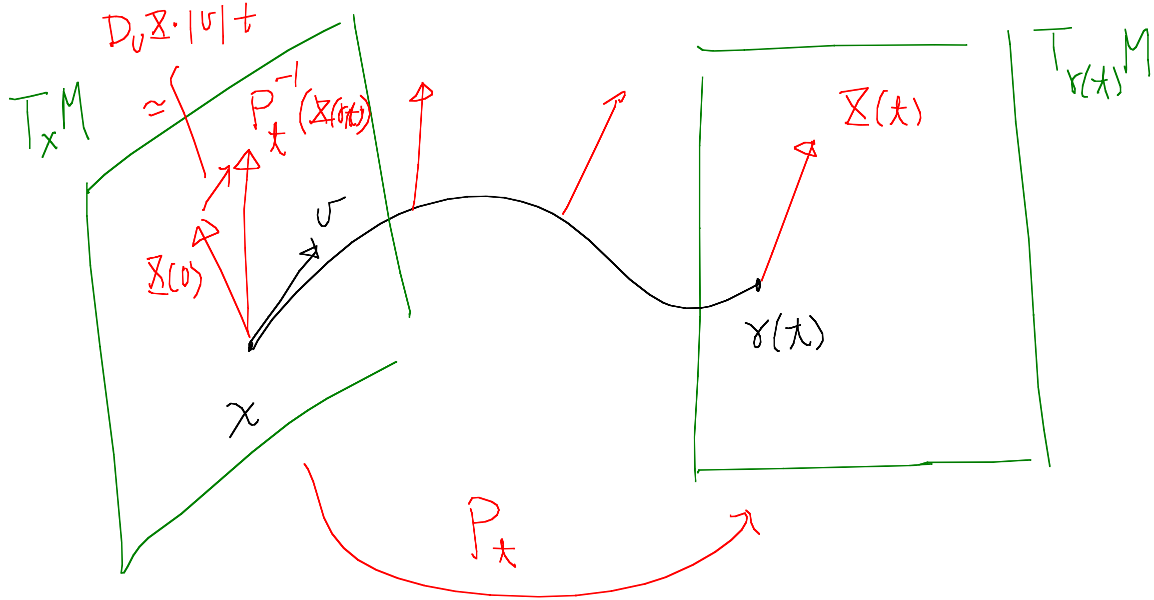
<u>Thm</u> : $\forall v \in T_x M$ & $X \in \Gamma(TM)$,

$$D_v X = \frac{d}{dt}\Big|_{t=0} P_t^{-1}\big(X(\gamma(t))\big) \qquad \left(\begin{array}{c}\text{for } D \\ \text{Levi-Civita}\end{array}\right)$$

where $\gamma : [0,1] \to M$ is a curve s.t.

$\gamma(0) = x$ & $\gamma'(0) = v$

$P_t : T_x M \to T_{\gamma(t)} M$ = parallel transport along $\gamma|_{[0,t]}$.




Pf: Let $\{e_i\}$ be an orthonormal basis of $T_x M$.

Define $e_i(t) = P_t e_i$

Then $\{e_i(t)\}$ is an o.n. basis of $T_{\gamma(t)} M$.

Write $X$ in terms of $\{e_i(t)\}$:

$$X(\gamma(t)) = \sum \hat{X}^i(t)\, e_i(t) \quad \text{for some } \hat{X}^i(t)$$

$$\Rightarrow \quad D_v X = \sum \frac{d\hat{X}^i}{dt}(0)\, e_i$$

And $$P_t^{-1}(X(\gamma(t)) = \sum \hat{X}^i(t)\, P_t^{-1}(e_i(t))$$

$$= \sum \hat{X}^i(t)\, e_i \quad \in T_x M$$

$$\Rightarrow \quad \left.\frac{d}{dt}\right|_{t=0} P_t^{-1}(X(\gamma(t))) = \sum \frac{d\hat{X}^i}{dt}(0)\, e_i = D_v X.$$

$\#$

## 2.3 Geodesic

Def: A curve $\gamma:[a,b]\to M$ is called a geodesic wrt the connection $D$ if $\gamma'(t)$ is parallel along $\gamma$.

In local coordinates $\{x^i\}$

$$\gamma(t) = (x^1(t), \dots, x^n(t))$$

$$\Rightarrow \quad \gamma'(t) = \sum \frac{dx^i}{dt}(t) \left.\frac{\partial}{\partial x^i}\right|_{\gamma(t)}$$

Hence

$$D_{\gamma'(t)}\gamma'(t) = \sum_k \left[\frac{d}{dt}\left(\frac{dx^k}{dt}\right) + \Gamma^k_{ij}(\gamma(t))\frac{dx^i}{dt}\frac{dx^j}{dt}\right]\frac{\partial}{\partial x^k}$$

$\therefore$ $\gamma$ is a geodesic (wrt $D$) $\Leftrightarrow$ $D_{\gamma'}\gamma' = 0$

$$\Leftrightarrow \quad \frac{d^2x^k}{dt^2} + \Gamma^k_{ij}(x^1,\dots,x^n)\frac{dx^i}{dt}\frac{dx^j}{dt} = 0 \quad \forall\, k = 1,\dots,n$$

which is a <u>non-linear</u> ODE system for $(x^1(t),\dots,x^n(t))$.

ODE theory $\Rightarrow$

<u>Lemma:</u> $\forall$ connection $D$ on $M$;

$\forall\, v \in T_xM$

$\Rightarrow$ $\exists!$ geodesic $\gamma(t)$ wrt $D$ on <u>some interval</u> $(-\varepsilon,\varepsilon)$

s.t. $\gamma(0) = x$ and $\gamma'(0) = v$.

Note: If $D$ is Levi-Civita connection of $g$.

Then $\forall$ geodesic $\gamma$ of $D$, we have

$$\frac{d}{dt}\langle \gamma', \gamma' \rangle = \langle D_{\gamma'}\gamma', \gamma' \rangle + \langle \gamma', D_{\gamma'}\gamma' \rangle = 0$$

$\Rightarrow$ $|\gamma'(t)|$ is a constant.

## 2.4 Induced connection

Let $M$ = Riemannian manifold

$N$ = differentiable manifold

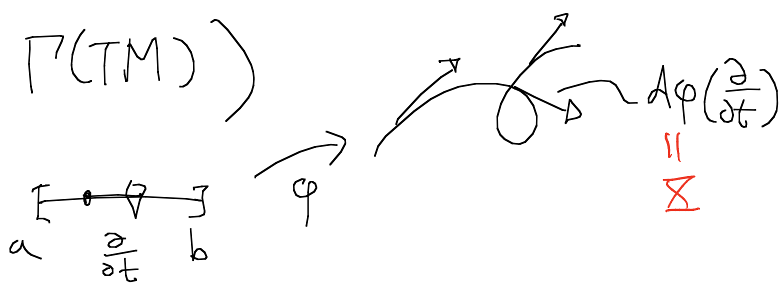
and $\varphi : N \to M$ $C^\infty$ map

Def: A map $X : N \to TM$ is called a <u>vector field along</u> $\varphi$ if $\forall x \in N$, $X(x) \in T_{\varphi(x)}M$.

$$\begin{tikzcd} & TM \arrow[d] \\ N \arrow[ur, "X"] \arrow[r, "\varphi"'] & M \arrow[ul, phantom, "\circlearrowleft"] \end{tikzcd}$$

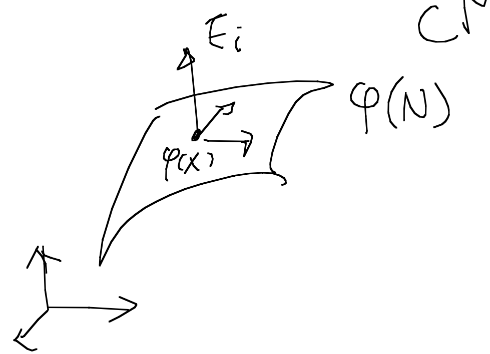
eg: $\overset{Y}{\cancel{X}} \in \Gamma(TN)$, $X = d\varphi(\overset{Y}{\cancel{X}})$ is a vector field along $\varphi$

(but not necessary $\in \Gamma(TM)$)



<u>Note</u>: If $v \in T_xN$, and $\{E_i\}_{i=1}^n$ is "frame field" in a nbd $V$ of $\varphi(x) \in M$

(ie $\{E_i(p)\}$ is a basis of $T_pM$ $\forall p$ in $V$, ($E_i(p)$ smooth in $p$))




Then $\forall x \in \varphi^{-1}(V) \subset N$

$$X(x) = \sum X^i(x) E_i(\varphi(x)) \in TM, \text{ for some functions } X^i(x) \text{ on } N.$$

Define

$$\tilde{D}_v X = \sum \left[ v(X^i)(x) E_i(\varphi(x)) + X^i(x) D_{d\varphi(v)} E_i \right]$$

where $D$ = Levi-Civita connection $M$

Fact: $\tilde{D}_v X$ is well-defined (indep of the choice of $\{E_i\}$)

<u>Def</u>: • $\tilde{D}$ is called the <u>induced connection</u>

• $\forall\ V \in \Gamma(TN)$, $X$ = vector field along $\varphi$

$$(\tilde{D}_V X)(x) \overset{\text{def}}{=\!=} \tilde{D}_{V(x)} X$$

<u>Facts</u>: If $D$ = Levi-Civita on $M$, then

• $\forall\ X, Y \in \Gamma(TN)$

$$\tilde{D}_X d\varphi(Y) - \tilde{D}_Y d\varphi(X) - d\varphi([X,Y]) = 0$$

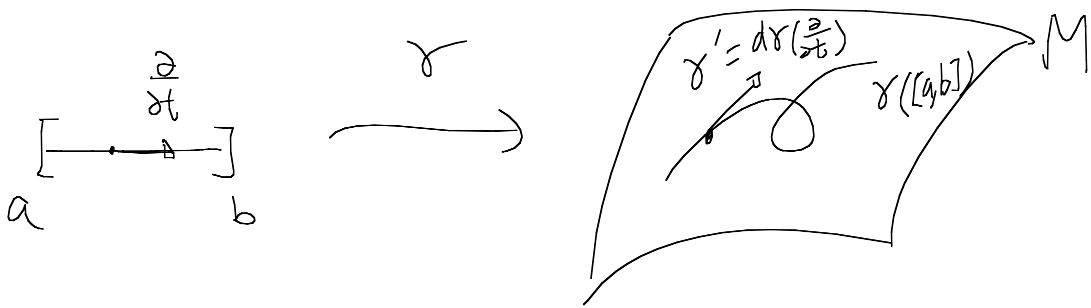
$d\varphi([X,Y]) = [d\varphi(X), d\varphi(Y)]$

- $\forall$ $V, W$ vecta fields along $\varphi$ & $u \in T_x N$,

then
$$u\langle V, W\rangle = \langle \tilde{D}_u V, W\rangle + \langle V, \hat{D}_u W\rangle$$

Note: If $\gamma : [0,1] \to M$ is a smooth curve (not necessary embedded) then

$$\gamma' = d\gamma\left(\frac{\partial}{\partial t}\right) \text{ is vecta field along } \gamma$$



$$\begin{tikzcd} & TM \arrow[d] \\ {[a,b]} \arrow[ur, "d\gamma(\frac{\partial}{\partial t})"] \arrow[r, "\gamma"'] \arrow[ur, phantom, "\circlearrowleft", shift right=3] & M \end{tikzcd}$$

We define $D_{\gamma'}\gamma' \overset{\text{def}}{=\!=} \tilde{D}_{\frac{\partial}{\partial t}}\gamma'$.

(Check: If $\gamma$ is embedded, this definition coincides with the previous one.)

$\therefore$ Geodesic (& $P^{\gamma}$) can be defined for any smooth curve.

# Ch3 Covariant derivative, Curvature Tensor

## 3.1 Covariant derivative of tensors

Fact: Let $\varphi : V \to W$ be an isomorphism between vector spaces, then $\varphi$ can be extended to an isomorphism between the tensor algebras:

$$\tilde{\varphi} : \bigoplus_{r,s} T^{r,s} V \to \bigoplus_{r,s} T^{r,s} W,$$

where $T^{r,s} V = (\underbrace{V \otimes \cdots \otimes V}_{r}) \otimes (\underbrace{V^* \otimes \cdots \otimes V^*}_{s}),$

$V^*$ = dual of $V$.

In fact, we can first define

$$\begin{array}{ccc} \varphi^*: W^* & \longrightarrow & V^* \\ \cup & & \cup \\ \alpha & \longmapsto & \varphi^*(\alpha) \end{array} \quad \text{by} \quad \boxed{\varphi^*(\alpha)(v) = \alpha(\varphi(v))}$$

Then $\varphi =$ isom $\Rightarrow \varphi^*$ isom

i.e. $(\varphi^*)^{-1} : V^* \to W^*$ exists

Hence we can define

$\forall \; v_1 \otimes \cdots \otimes v_r \otimes \alpha^1 \otimes \cdots \otimes \alpha^s \in T^{r,s} V$,

$$\tilde{\varphi}(v_1 \otimes \cdots \otimes v_r \otimes \alpha^1 \otimes \cdots \otimes \alpha^s)$$
$$= \varphi(v_1) \otimes \cdots \otimes \varphi(v_r) \otimes (\varphi^*)^{-1}(\alpha^1) \otimes \cdots \otimes (\varphi^*)^{-1}(\alpha^s) \in T^{r,s} W.$$

Finally, extend $\tilde{\varphi}$ to all $\bigoplus_{r,s} T^{r,s} V$ by linearity and can be checked that $\tilde{\varphi}$ is an isomorphism.

<u>Def</u>: Let $M$ = Riemannian manifold, $x \in M$, $v \in T_x M$,

$\gamma$ = curve with $\gamma(0) = x$, $\gamma'(0) = v$.

Then $\forall$ tensor field $K$ on $M$, we define the <u>covariant derivative</u> of $K$ <u>wrt $v$</u> by

$$D_v K = \frac{d}{dt}\bigg|_{t=0} \left(\tilde{P}^{\gamma}_t\right)^{-1} \left( K(\gamma(t)) \right)$$

where $\tilde{P}^{\gamma}_t : \bigoplus_{r,s} T^{r,s}(T_x M) \longrightarrow \bigoplus_{r,s} T^{r,s}(T_{\gamma(t)} M)$

is the extension of the parallel transport

$P_t^{\gamma} : T_x M \longrightarrow T_{\gamma(t)} M$ wrt Levi-Civita connection.

Caution: We need to check $D_v K$ does not depend on $\gamma$.

Properties:

(1) If $K$ is a $(r,s)$-tensor, then $D_v K$ is also a $(r,s)$-tensor.

(2) $D_v$ is a derivation on the tensor algebra:

$$D_v(K_1 \otimes K_2) = (D_v K_1) \otimes K_2 + K_1 \otimes (D_v K_2)$$

(3) $D_v$ commutes with "contractions".

Def (of contraction) The contractions $C_{pq}$, $p = 1, \dots, r$; $q = 1, \dots, s$

are linear maps

$$C_{pq} : \left( \otimes^{r} TM \right) \otimes \left( \otimes^{s} T^{*}M \right) \to \left( \otimes^{r-1} TM \right) \otimes \left( \otimes^{s-1} T^{*}M \right)$$

defined by

$$C_{pq}\left( v_1 \otimes \cdots \otimes v_r \otimes \alpha^1 \otimes \cdots \otimes \alpha^s \right)$$

$$= \alpha^q(v_p) \; v_1 \otimes \cdots \otimes \hat{v}_p \otimes \cdots \otimes v_r \otimes \alpha^1 \otimes \cdots \otimes \widehat{\alpha^q} \otimes \cdots \otimes \alpha^s$$

omitted

egs: Fa $C_{11} : TM \otimes T^{*}M \longrightarrow \mathbb{R} \; \left( \simeq \otimes^0 TM \otimes \otimes^0 T^{*}M \right)$

takes $\frac{\partial}{\partial x^i} \otimes dx^j \longmapsto C_{11}\left( \frac{\partial}{\partial x^i} \otimes dx^j \right) = dx^j\left( \frac{\partial}{\partial x^i} \right) = \delta^j_i$

Fa $C_{11} : TM \otimes \otimes^2 T^{*}M \longrightarrow T^{*}M$

takes $\frac{\partial}{\partial x^i} \otimes dx^{j_1} \otimes dx^{j_2} \mapsto C_{11}\left(\frac{\partial}{\partial x^i} \otimes dx^{j_1} \otimes dx^{j_2}\right)$

$$= dx^{j_1}\left(\frac{\partial}{\partial x^i}\right) dx^{j_2} = \delta^{j_1}_{i}\, dx^{j_2} \in T^*M$$

Property (3) means if $\mathcal{C} = C_{pq}$ is a contraction, then

$$\boxed{D_v(\mathcal{C}K) = \mathcal{C}(D_v K)}.$$

Pf: (1) is clear.

(2) We do a special case only. The general case can be proved similarly.

Suppose $K = \underline{X} \otimes \omega \otimes \rho \in TM \otimes (\overset{2}{\otimes} T^*M)$

i.e. $X$ = vector field,

$\omega, \rho$ = 1-forms (ie linear combinations of $dx^i$)

Then we need to prove that

$$D_v K = (D_v X) \otimes \omega \otimes \rho + X \otimes D_v \omega \otimes \rho + X \otimes \omega \otimes D_v \rho$$

Let $\{e_1(t), \dots, e_n(t)\}$ be parallel vector fields along $\gamma$

s.t. $\{e_i(t)\}$ forms a basis of $T_{\gamma(t)}M$.

i.e. $D_{\gamma'} e_i(t) \equiv 0$.

Then $\forall t$, $\exists$ dual basis $\{\alpha^1(t), \dots, \alpha^n(t)\}$ of $T^*_{\gamma(t)}M$,

i.e. $\alpha^i(t)(e_j(t)) = \delta^i_j$, $\forall t$.

By definition of $\widetilde{P}_t$, we see that

$$\widetilde{P}_t(\alpha^i(0)) \overset{\text{def}}{=} (P_t^*)^{-1}(\alpha^i(0))$$

$$\Longleftrightarrow \quad P_t^*\left(\widetilde{P}_t(\alpha^i(0))\right) = \alpha^i(0)$$

$$\Longleftrightarrow \quad P_t^*\left(\widetilde{P}_t(\alpha^i(0))\right)(e_j(0)) = \alpha^i(0)(e_j(0)) = \delta^i_j \quad \forall j$$

$$\Longleftrightarrow \quad \widetilde{P}_t(\alpha^i(0))\left(P_t(e_j(0))\right) = \delta^i_j \quad \forall j$$

$$\Longleftrightarrow \quad \widetilde{P}_t(\alpha^i(0))(e_j(t)) = \delta^i_j \quad \forall j$$

$$\Longleftrightarrow \quad \widetilde{P}_t(\alpha^i(0)) = \alpha^i(t)$$

i.e. $\{\alpha^i(t)\}$ are all parallel.

Write
$$\begin{cases} X(t) = X(\gamma(t)) = \sum_i X^{i}(t) e_i(t) \\ \omega(t) = \omega(\gamma(t)) = \sum_j \omega_j(t)\, \alpha^{j}(t) \\ \rho(t) = \rho(\gamma(t)) = \sum_l \rho_l(t) \alpha^{l}(t) \end{cases}$$

Then $$K(t) = \sum_{i,j,l} X^{i}(t)\, \omega_j(t) \rho_l(t)\; e_i(t) \otimes \alpha^{j}(t) \otimes \alpha^{l}(t)$$

$$\Rightarrow \tilde{P}_t^{-1} K(t) = \sum_{i,j,l} X^{i}(t)\, \omega_j(t) \rho_l(t)\; e_i(0) \otimes \alpha^{j}(0) \otimes \alpha^{l}(0)$$

$$\Rightarrow D_v K = \frac{d}{dt}\Big|_{t=0} \tilde{P}_t^{-1} K(t)$$

$$= \sum_{i,j,l} \left( \frac{dX^{i}}{dt} \omega_j \rho_l + X^{i} \frac{d\omega_j}{dt} \rho_l + X^{i} \omega_j \frac{d\rho}{dt} \right) e_i(0) \otimes \alpha^{j}(0) \otimes \alpha^{l}(0).$$

Similarly

$$\begin{cases} D_v X = \sum_i \frac{dX^i}{dt} e_i(0) \\ D_v \omega = \sum_j \frac{d\omega_j}{dt} \alpha^j(0) \\ D_v \rho = \sum_l \frac{d\rho_l}{dt} \alpha^l(0) \end{cases}$$

$$\Rightarrow \quad D_v K = D_v X \otimes \omega \otimes \rho + X \otimes D_v \omega \otimes \rho + X \otimes \omega \otimes D_v \rho$$

This proves (2).

Pf of (3) We do the special case that

$$K = X \otimes \omega \otimes \rho \in TM \otimes (\otimes^2 T^*M) \quad \&$$

$$\mathcal{C} = C_{12} : TM \otimes (\otimes^2 T^*M) \to T^*M$$

In this case $\quad \mathcal{C}K = \mathcal{C}(X \otimes \omega \otimes \rho)$

$$= \rho(X)\,\omega \in T^*M$$

$$\mathcal{C}(D_v K) = \mathcal{C}\left(D_v X \otimes \omega \otimes \rho + X \otimes D_v \omega \otimes \rho + X \otimes \omega \otimes D_v \rho\right)$$

$$= \rho(D_v X)\,\omega + \rho(X) D_v \omega + (D_v \rho)(X)\,\omega$$

$\therefore$ We need to show that

$$D_v(\rho(X)\,\omega) = \rho(D_v X)\,\omega + \rho(X) D_v \omega + (D_v \rho)(X)\,\omega.$$

Note that $\rho(X) = \left(\sum_l \rho_l \alpha^l(t)\right)\left(\sum_i X^i e_i(t)\right)$

$$= \sum_{l,i} \rho_l X^i \delta_i^l = \sum_i \rho_i X^i$$

$$\rho(D_v X) = \sum_i \rho_i \frac{dX^i}{dt}$$

$$(D_v \rho)(X) = \sum_i \frac{d\rho_i}{dt} X^i$$

$$\Rightarrow \quad \rho(D_v X)\,\omega + \rho(X) D_v \omega + (D_v \rho)(X)\,\omega$$

$$= \left[ \left(\rho_i \frac{dX^i}{dt}\right) \omega_j + (\rho_i X^i) \frac{d\omega_j}{dt} + \left(\frac{d\rho_i}{dt} X^i\right) \omega_j \right] \alpha^j(0)$$

and $D_v(\rho(X)\omega) = D_v\left( (\rho_i X^i)\, \omega_j \alpha^j(t) \right)$

$$= \frac{d}{dt}\Big|_{t=0} \left[ \left[ (\rho_i X^i) \omega_j \right] \alpha^j(0) \right]$$

$$= \rho(D_v X)\,\omega + \rho(X) D_v \omega + (D_v \rho)(X)\,\omega$$

$\#$

Note: • One can define $D_v \rho$ by

$$D_v \left[ \mathcal{C}(X \otimes \rho) \right] = \mathcal{C}\left( D_v (X \otimes \rho) \right)$$

i.e. $v(\rho(X)) = C(D_v X \otimes \rho + X \otimes D_v \rho)$

$= \rho(D_v X) + (D_v \rho)(X)$

i.e. $\boxed{(D_v \rho)(X) = v(\rho(X)) - \rho(D_v X) \quad \forall X \in \Gamma(TM)}$

- This also shows that $D_v K$ does not depend on $\gamma$ (since the RHS does not depend on $\gamma$).

Def: Let $K$ = tensor field on $M$,

$X$ = vector field on $M$

Then we define $(D_X K)(x) \overset{\text{def}}{=} D_{X(x)} K$, $\forall x \in M$.

<u>Note</u>: By linearity of $D_X K$ in $X$, on can define

$$DK \in (\otimes^r TM) \otimes (\otimes^{s+1} T^*M) \quad \left(\text{for } K \in (\otimes^r TM) \otimes (\otimes^s T^*M)\right)$$

by requiring

Ex: think careful about this!

$$DK(\omega^1 \otimes \cdots \otimes \omega^r \otimes X_1 \otimes \cdots \otimes X_s \otimes X)$$

$$\underset{=}{\underline{\text{def}}} \; (D_X K)(\omega^1 \otimes \cdots \otimes \omega^r \otimes X_1 \otimes \cdots \otimes X_s)$$

[<u>Caution</u>: Some authors put

$$DK(\omega^1 \otimes \cdots \otimes \omega^r \otimes X \otimes X_1 \otimes \cdots \otimes X_s) = (D_X K)(\cdots\cdots)$$ ]

<u>Note</u>: If $K = f \in T^{(0,0)}M \cong C^\infty(M)$.

Then $Df = df$ the usual differential of $f$.

<u>Def</u>: For $n \geq 0$, we define

$$D^{n+1} K = D(D^n K)$$

<u>Note</u>: $D^2 K(\cdots, X, Y) \neq D_Y(D_X K)(\cdots)$ in general.

<u>eg</u>: Let $K = f \in C^\infty(M)$

Then
$$
\begin{aligned}
D^2 f(X, Y) &= \left(D(df)\right)(X, Y) \\
&= \left(D_Y(df)\right)(X) \\
&= Y(df(X)) - df(D_Y X) \\
&= YXf - (D_Y X) f \\
&\neq D_Y(D_X f)
\end{aligned}
$$

(by definition $D_Y(D_X f) = D_Y(Xf) = Y(Xf) = YXf$ )

Note: $D^2 f(X,Y) = YXf - (D_Y X)(f)$

$D^2 f(Y,X) = XYf - (D_X Y)(f)$

$\Rightarrow \quad D^2 f(X,Y) - D^2 f(Y,X) = -[X,Y]f + (D_X Y - D_Y X) f$

$= T(X,Y) f$

torsion tensor

$\therefore$ D symmetric (torsion free) $\Leftrightarrow$ $D^2 f$ is symmetric

In this case, $D^2 f$ is called the Hessian of $f$.

From now on, we assume $M$ has a Riemannian metric $g$, and $D$ = Levi-Civita connection of $g$.

Therefore $D^2 f$ is always symmetric $\forall f \in C^\infty(M)$.

Def: $\forall S \in \otimes^2 T^*M$, we define $\operatorname{tr} S \in C^\infty(M)$ the trace of $S$, by

$$\operatorname{tr} S(x) = \sum_i S(e_i, e_i)$$

where $\{e_i\}$ is an orthonormal basis of $T_xM$.

Note: • $\operatorname{tr} S$ is well-defined, i.e. independent of the choice of o.n. basis $\{e_i\}$.

- $(\operatorname{tr} S)(x)$ is smooth in $x$

(Pf: Ex)

<u>Def</u>: let $(M, g)$ = Riemannian manifold

$D$ = Levi-Civita connection of $g$.

Then the <u>Laplace operator</u>, <u>Laplacian</u> or <u>Laplace-Beltrami</u> operator

$$\Delta : C^\infty(M) \longrightarrow C^\infty(M)$$

is defined by $\Delta f = \operatorname{tr} D^2 f$.

<u>Ex</u>: Prove that in local coordinates $(x^1, \cdots, x^n)$

$$\Delta f = \frac{1}{\sqrt{G}} \sum_j \frac{\partial}{\partial x^j}\left( \sum_i g^{ij} \sqrt{G} \frac{\partial f}{\partial x^i} \right)$$

where $G = \det(g_{ij})$, $g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$ & $(g^{ij}) = (g_{ij})^{-1}$