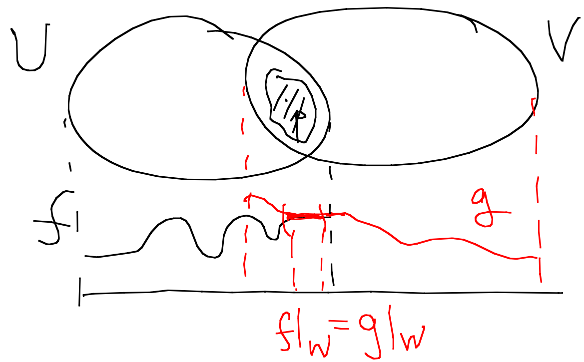
1.5 Tangent vectors as derivations

Let M be a smooth manifold, $p \in M$, consider C^∞ functions defined in a neighborhood of p. Then we can define an equivalence relation : $f: U \to \mathbb{R} \sim g: V \to \mathbb{R}$

$(p \in U,\ p \in V)$

$\Leftrightarrow \exists$ nbd. $W \subset U \cap V$ of p s.t. $f|_W = g|_W$



Def: The equivalence classes for this relation are the germs of C^∞ functions at p. The space of germs of C^∞ functions at p is denoted $\mathcal{C}_p^\infty(M)$.

Similarly, we can define $\mathcal{C}_p^0(M)$, $\mathcal{C}_p^k(M)$ & $\mathcal{C}_p^\omega(M)$ the germs of continuous, C^k, & (real) analytic functions respectively at p.

Remarks: • Space of functions has linear structure (with product structure)

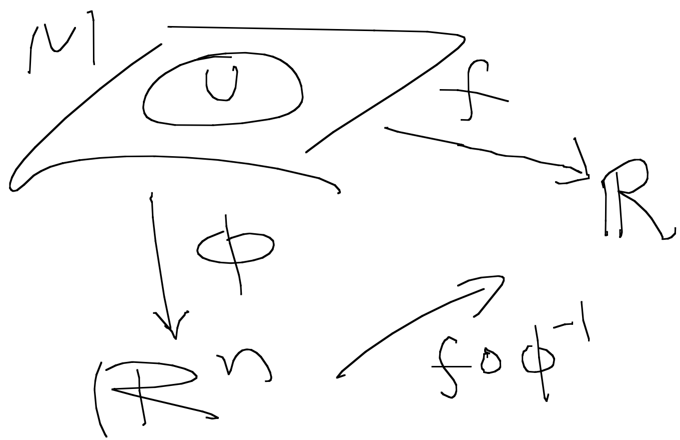
$\Rightarrow$ corresponding space of germs is a vector space (with product structure).

• If M is a C^k manifold $(0 \le k \le \infty)$

then $\mathcal{C}^k_p(M) \cong \mathcal{C}^k_0(\mathbb{R}^n)$ (vector space isomorphism)

Pf: germ of f $\longleftrightarrow$ germ of $f \circ \phi^{-1}$

for a chart (U, ϕ)



Def: A derivation on $\mathcal{C}^k_p(M)$ is a linear map

$\delta : \mathcal{C}^k_p(M) \to \mathbb{R}$ such that $\forall\, f, g \in \mathcal{C}^k_p(M)$

$$\delta(fg) = f(p)\,\delta(g) + g(p)\,\delta(f)$$

(where fg = product of the germs f, g : How to define? (Ex.))

Notation: We denote the set of derivations on $\mathcal{C}^k_p(M)$ by $\mathcal{D}^k_p(M)$ or $\mathcal{D}_p(M)$ if k is clear.

Thm: Any derivation of $\mathcal{C}^\infty_0(\mathbb{R}^n)$ can be written as

$$\delta(f) = \sum_{j=1}^{n} \delta(x^j)\, \frac{\partial f}{\partial x^j}(0)$$

this f is a function representing the germ f.

Hence $\dim(\mathcal{D}_0^\infty(\mathbb{R}^n)) = n$.

(where x^j = germ of the coordinate function $x^j : \mathbb{R}^n \to \mathbb{R}$
$$\begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix} \mapsto x^j \quad)$$

Pf: $\forall$ germ $f \in \mathcal{C}_0^\infty(\mathbb{R})$, f is represented by a C^∞ function in a nbd. of 0. We denote this function by f again. Then

$$f(x) - f(0) = \int_0^1 \frac{d}{dt} f(tx)\, dt$$
$$= \int_0^1 \sum_{j=1}^n \frac{\partial f}{\partial x^j}(tx)\, x^j\, dt$$
$$= \sum_{j=1}^n x^j h_j(x)$$

where $h_j(x) = \int_0^1 \frac{\partial f}{\partial x^j}(tx)\,dt$ is C^∞.

Then $\delta(f) = \delta(f - f(0))$ since $\delta(\text{const.}) = 0$ (Ex.)

$$= \sum_{j=1}^n \delta(x^j h_j)$$

$$= \sum_{j=1}^n x^j(0)\,\delta(h_j) + h_j(0)\,\delta(x^j)$$

$$= \sum_{j=1}^n \delta(x^j)\,\frac{\partial f}{\partial x^j}(0) \quad \#$$

Lemma $\forall \xi \in T_pM$, $L_\xi(f) \overset{\text{def}}{=\!=} (D_pf)(\xi)$ $\forall f \in C_0^\infty(M)$

Then $L_\xi \in \mathcal{D}_p(M)$

(Pf: Easy Ex) where D_pf is the differential of f defined similarly as in Diff. Geom. using def 1 of vectors.

<u>Thm</u>: $T_pM \longrightarrow \mathcal{D}_p(M)$ is an isomorphism

$\xi \longmapsto L_\xi$ (between vector spaces)

<u>Pf</u>: • $\xi \mapsto L_\xi$ is clearly linear

• Let (U, ϕ) be a chart for M around p with $\phi(p) = 0$ ($\in \mathbb{R}^n$).

Then ξ can be represented by

$$\xi = (U, \phi, v) \quad \text{with } v \in T_0\mathbb{R}^n \cong \mathbb{R}^n.$$

$\Rightarrow$ $\forall$ C^∞ function f in a nbd around p.

$$L_\xi f = D_0(f \circ \phi^{-1})(v) \quad \text{(EX!)}$$

$$= \sum_{j=1}^{n} v^j \frac{\partial}{\partial x^j}(f \circ \phi^{-1})(0), \quad \text{where } v = (v^1, \cdots, v^n)$$

If $\xi \in \ker(\xi \mapsto L_\xi)$, then $\forall f$

$$0 = \sum_{j=1}^{n} v^j \frac{\partial}{\partial x^j}(f \circ \phi^{-1})(0)$$

$\Rightarrow v^j = 0, \forall j \Rightarrow \xi = 0$. $\therefore \ker(\xi \mapsto L_\xi) = 0$

- Finally, $\forall \delta \in \mathcal{D}_p(M) \cong \mathcal{D}_0(\mathbb{R}^n)$, previous lemma

$$\Rightarrow \delta(f) = \sum_{j=1}^{n} \delta(x^j) \frac{\partial}{\partial x^j}(f \circ \phi^{-1})(0)$$

$$\therefore \delta = L_\xi \text{ for } \xi = \left[\left(U, \phi, \begin{pmatrix} \delta(x^1) \\ \vdots \\ \delta(x^n) \end{pmatrix}\right)\right] \in T_pM$$

$$\Rightarrow \mathrm{Im}(\xi \mapsto L_\xi) = \mathcal{D}_p(M)$$

\#

Remark: In particular, we have $\dim T_pM = n$ with basis corresponds to $\left\{ \frac{\partial}{\partial x^j}\Big|_0 \right\}$ in local coordinates

$$\left\{ \frac{\partial}{\partial x^j}\Big|_0 \right\} \subset \mathcal{D}_0(\mathbb{R}^n)$$

(i.e. a germ s.t $\begin{pmatrix} \delta(x^1) \\ \vdots \\ \delta(x^n) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j\text{th place}$)

Convention: If (U, ϕ) is a chart around p, and $(x^1, \cdots, x^n)$ are the corresponding coordinate functions

$$x^j : U \xrightarrow{\phi} \mathbb{R}^n \xrightarrow{\pi_j} \mathbb{R}.$$

We denote $\left(\frac{\partial}{\partial x^j}\right)_p (f) \overset{\text{def}}{=} \frac{\partial (f \circ \phi^{-1})}{\partial x^j} (\phi(p))$

In this notation

$$L\xi = \sum_{j=1}^{n} v^j \left(\frac{\partial}{\partial x^j}\right)_p \quad \text{for } \xi = [(U, \phi, v)] \in T_pM$$

Hence $\left(\frac{\partial}{\partial x^j}\right)_p$ can be regarded as a vector in T_pM;

$\Rightarrow$ $\frac{\partial}{\partial x^j}$ is an vector field on $U \subset M$.

If $X^1, \dots, X^n$ are smooth functions, then

$$X = \sum_{j=1}^{n} X^j \frac{\partial}{\partial x^j} \quad \text{is a vector field on } U.$$

corresponds to $L_X : C^\infty(U) \to C^\infty(U)$ defined by

$$(L_X f)(p) = \sum_{j=1}^{n} X^j(p) \left(\frac{\partial f}{\partial x^j}\right)_p$$

Thm: The map $X \mapsto L_X$ is an isomorphism between the vector spaces $\Gamma(TM)$ and $\mathcal{D}(M)$, where $\mathcal{D}(M)$ = set of derivations δ on M which are defined by requiring

(i) $\delta : C^\infty(M) \to C^\infty(M)$ linear;

(ii) $\delta(fg) = f\delta(g) + g\delta(f)$

(Pf: Omitted)

(Caution: Analog statement for complex manifold is not true. Since we need to use cut-off functions to reduce to coordinate systems.)

Note: If $\delta_1, \delta_2 \in \mathcal{D}(M)$, then $\delta_1 \circ \delta_2 \notin \mathcal{D}(M)$

Lemma: If $\delta_1, \delta_2 \in \mathcal{D}(M)$, then

$$\delta_1 \circ \delta_2 - \delta_2 \circ \delta_1 \in \mathcal{D}(M)$$

Pf (Exercise)

Def: Let X, Y be vector fields on M. Then $[X, Y]$, the bracket of X & Y, is the vector field corresponding to the derivation $L_X \circ L_Y - L_Y \circ L_X$

(i.e. $L_{[X,Y]} = L_X \circ L_Y - L_Y \circ L_X$)

Local formula for $[X, Y]$

If $X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i}$, $Y = \sum_{j=1}^{n} Y^j \frac{\partial}{\partial x^j}$ in some local coordinates

then

$$L_X f = \sum_i X^i \frac{\partial f}{\partial x^i}$$

$$\Rightarrow \quad L_Y L_X f = \sum_{i,j} Y^j X^i \frac{\partial^2 f}{\partial x^j \partial x^i} + Y^j \frac{\partial X^i}{\partial x^j} \frac{\partial f}{\partial x^i}$$

Similar formula for $L_X L_Y f$.

$$\Rightarrow (L_X L_Y - L_Y L_X) f = \sum_i \left(\sum_j -Y^j \frac{\partial X^i}{\partial x^j} + X^j \frac{\partial Y^i}{\partial x^j} \right) \frac{\partial f}{\partial x^i}$$

$$\Rightarrow \boxed{\begin{gathered} [X, Y] = \sum_i Z^i \frac{\partial}{\partial x^i} \quad \text{with} \\ Z^i = \sum_j \left(-Y^j \frac{\partial X^i}{\partial x^j} + X^j \frac{\partial Y^i}{\partial x^j} \right). \end{gathered}}$$

<u>Lemma</u> (Jacobi identity) For vector fields X, Y, Z,
$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (Pf: Ex)

1.6 Vecter Bundles and Tensors

Def: Let E & B be 2 smooth manifolds and

$\pi : E \to B$ be a smooth map.

(π, E, B) is a vecter bundle of rank n,

if
- π is surjective;
- $\exists$ open covering $(U_i)_{i \in \Lambda}$ of B, and

  diffeomorphisms $h_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^n$

  s.t. $\forall x \in U_i \quad h_i(\pi^{-1}(x)) = \{x\} \times \mathbb{R}^n$

(hence $\pi^{-1}(x)$ can be regarded as a vecter space.)

- and such that $\forall i, j \in \Lambda$, the diffeomorphism

$$h_i \circ h_j^{-1} : (U_i \cap U_j) \times \mathbb{R}^n \to (U_i \cap U_j) \times \mathbb{R}^n$$

are of the form

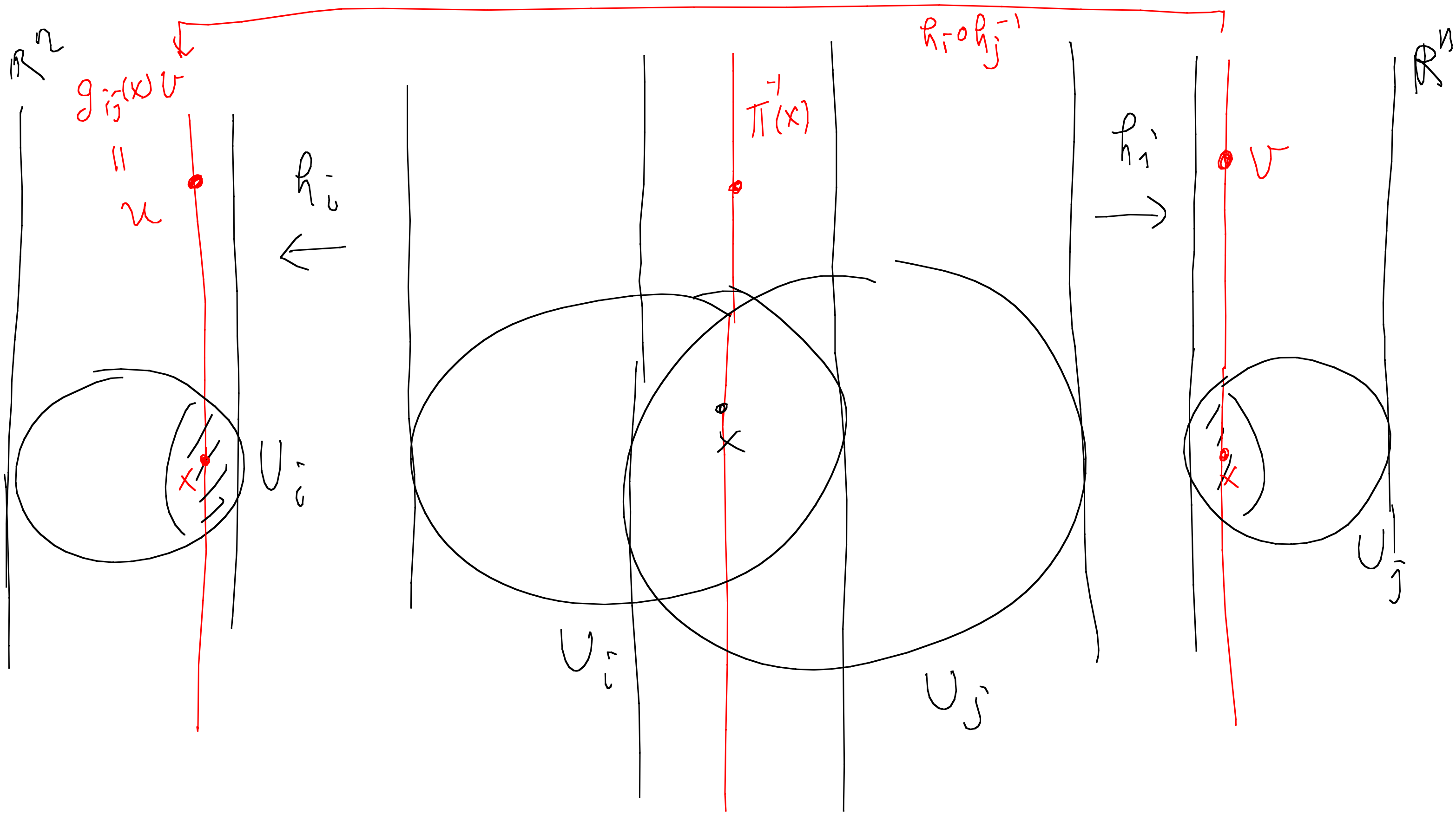
$$h_i \circ h_j^{-1}(x, v) = (x, g_{ij}(x) v)$$

where $g_{ij} : U_i \cap U_j \to GL(n, \mathbb{R})$

<u>Terminology</u>: E = <u>total space</u>, B = <u>base</u>.

$\mathbb{R}^n \cong \pi^{-1}(x)$ = <u>fibre</u>

h_i = <u>local trivialization</u>



$\mathbb{R}^n$
$g_{ij}(x)v$
$=$
u
h_i
U_i
$\pi^{-1}(x)$
x
$h_i \circ h_j^{-1}$
h_j
v
U_j
$\mathbb{R}^n$

eg: (Trivial Bundle): $\pi : M \times \mathbb{R}^n \longrightarrow M$
$$(x,v) \longmapsto x$$

eg: Tangent bundle of M: $TM = \coprod_{p \in M} T_pM$ (Exercise)

Def: (a) A vector bundle of rank n, $\pi : E \to B$, is trivial if $\exists$ a diffeomorphism
$$h : E \to B \times \mathbb{R}^n$$
s.t. $h : \pi^{-1}(x) \longrightarrow \{x\} \times \mathbb{R}^n$ is a vector space isomorphism.

(b) A (global) section of the bundle is a smooth map $s : B \longrightarrow E$ s.t. $\pi \circ s = \mathrm{id}$

$$\begin{tikzcd} E \arrow[d, shift right, "\pi"'] \\ B \arrow[u, shift right, "s"'] \end{tikzcd}$$

eg: vector field $X \in \Gamma(M) \ (= \Gamma(TM))$ is a section of the tangent bundle TM.

Tensor product

Def: Let E, F be 2 finite dimensional vector spaces, then $E \otimes F$, the tensor product of E & F, is defined as the vector space, unique up to isomorphism, such that

$\forall$ vector space G,

$$L(E \otimes F, G) \overset{\text{isom}}{\cong} L_2(E \times F, G)$$

(linear transformations from $E \otimes F$ to G) (bilinear maps from $E \times F$, G)

Remark: $\exists$ a bilinear map $\otimes : E \times F \longrightarrow E \otimes F$ such that if $\{e_i\}$ = basis of E &

$\{f_j\}$ = basis of F,

then $\{e_i \otimes f_j\}_{i,j}$ is a basis of $E \otimes F$

(Hence for $u = a^i e_i \in E$, $v = b^j f_j \in F$, then

$$u \otimes v = a^i b^j e_i \otimes f_j$$)

<u>Facts</u>: (1) If $E^* =$ dual of $E = L(E, \mathbb{R})$

$F^* =$ dual of F

then $E^* \otimes F^* \simeq L_2(E \times F, \mathbb{R})$

$\simeq L(E \otimes F, \mathbb{R}) = (E \otimes F)^*$

$$\alpha \otimes \beta \longmapsto \alpha \otimes \beta(u \otimes v) = \alpha(u)\beta(v)$$

(by $\alpha \otimes \beta \longmapsto \alpha \otimes \beta(u \otimes v) = \alpha(u)\beta(v)$.)

(2) If $\alpha \in L(E,E')$ & $\beta \in L(F,F')$

(E, E', F, F' are finite dim'l vector spaces)

then one can define

$$\alpha \otimes \beta \in L(E \otimes F, E' \otimes F')$$

by $(\alpha \otimes \beta)(u \otimes v) \overset{\text{def}}{=\!=} \alpha(u) \otimes \beta(v)$

[If $E' = \mathbb{R} = F'$, then $E' \otimes F' = \mathbb{R}$

$\Rightarrow L(E \otimes F, E' \otimes F') \cong L(E \otimes F, \mathbb{R})$ as in (1)]

(3) Given a vector bundle E (with fibers E_x, $x \in M$),

one can define the vector bundle E^*, $\otimes^p E$

(with fibers E_x^*, and $\otimes^p E_x$ respectively)

(4) Given 2 vector bundles E, F (with fibers E_x, F_x) with the same base manifold M, we can define the vector bundle $E \otimes F$ over M with fiber $E_x \otimes F_x$.

eg: Starting from TM, we can define the cotangent bundle T^*M of M, and the (p,q)-tensor bundle $(\overset{p}{\otimes} TM) \otimes (\overset{q}{\otimes} T^*M)$ of M

Def: A (p,q)-tensor (field), or more precisely p times contravariant & q times covariant tensor, on M is a smooth section of the bundle

$$\left(\bigotimes^p TM\right) \otimes \left(\bigotimes^q T^*M\right).$$

<u>Note</u>: For $f: M \to \mathbb{R}$ smooth, we can define

$$df \in \Gamma(T^*M) \text{ by } df(X) = L_X f, \quad \forall X \in \Gamma(M)$$

$$(Tf = Df) \qquad\qquad (= Xf)$$

Then $\{dx^j\}_{j=1}^n$ is a dual basis to $\left\{\frac{\partial}{\partial x^i}\right\}_{j=1}^n$

$$\left(dx^j\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i}(x^j) = \delta_i^j \right)$$

at each point in a coordinate system with coordinate functions $x^1, \dots, x^n$.

Therefore

$$\left\{ \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_p}} \otimes dx^{i_1} \otimes \cdots \otimes dx^{i_q} \right\}$$

forms a local basis for $\left(\bigotimes^p TM\right) \otimes \left(\bigotimes^q T^*M\right)$.

$\Rightarrow$ in coordinates, a (p,q)-tensor (field) can be written as

$$T = T^{j_1 \cdots j_p}_{\ \ i_1 \cdots i_q} \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_p}} \otimes dx^{i_1} \otimes \cdots \otimes dx^{i_q}.$$

1.7 Partitions of unity

Recall that all manifolds in this course are supposed to have the property that "partitions of unity" is always possible.

That is:

for any $\{U_i\}_{i\in\Lambda}$ = open cover of M,

$\exists$ locally finite open cover $\{V_k\}_{k\in\Lambda'}$ and

a family $\{\varphi_k\}_{k\in\Lambda'}$ of real smooth functions

on M such that

- $\{V_k\}_{k\in\Lambda'}$ is subordinate to $\{U_i\}_{i\in\Lambda}$

  (i.e. each $V_k \subset U_i$ for some i)

- $\operatorname{supp} \varphi_k \subset V_k$, $\varphi_k \geq 0$, $\sum_{k\in\Lambda'} \varphi_k(x) = 1$, $\forall x \in M$.

Here $\{V_k\}_{k\in\Lambda'}$ being locally finite means

$\forall\, x \in M$, $\exists$ open nbd W of x such that

$W \cap V_k = \emptyset$ except finite many k's.