# MATH5061 - Riemannian Geometry I - 2015/16

**Course Name:**
Riemannian Geometry I

**Teacher:**
Prof. Tom Yau Heng WAN

**Course Year:**
2015-16

**Term:**
2

## General Information

### Lecturer

- **WAN, Yau Heng Tom**
  - *Office: LSB215*
  - *Tel: 3943 7986*
  - *Email: tomwan@math.cuhk.edu.hk*

### Time and Venue

- *Lecture:* Monday 9:30-12:15pm; LSB222

## Course Description

This course is intended to provide a solid background in Riemannian Geometry. Topics include: affine connection, tensor calculus, Riemannian metric, geodesics, curvature tensor, completeness and some global theory. Students taking this course are expected to have knowledge in differential geometry of curves and surfaces.

## References

- S. Gallot, D. Hulin, J. Lafontaine, "Riemannian Geometry", 3rd Ed., Springer
- J. Jost, "Riemannian Geometry and Geometric Analysis", 2nd Ed., Springer
- I. Chavel, "Riemannian Geometry - A Modern Introduction", Cambridge
- 伍鴻熙，沈純理，虞言林，《黎曼幾何初步》，北京大學出版社

## Assessment Scheme

| | | |
|---|---|---|
| Final exam (Apr 25, 2016) | 100 | % |

## Honesty in Academic Work

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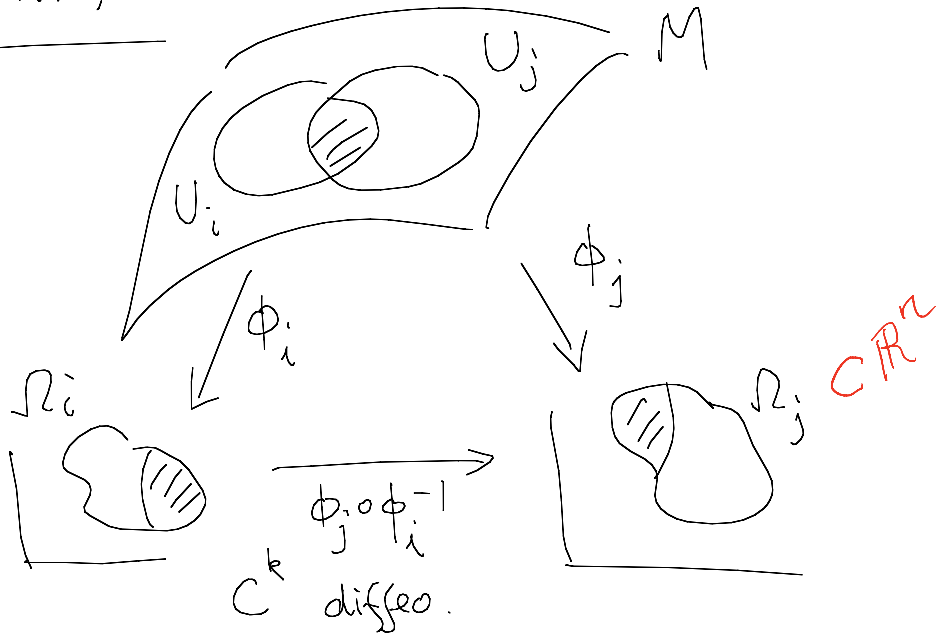
and thereby help avoid any practice that would not be acceptable.

# Ch 1 Differentiable Manifolds

## 1.1 Abstract Manifolds




Def: A $C^k$ atlas on a Hausdroff topological space $M$ is given by

(i) an open covering $U_i$, $i \in \Lambda$, of $M$;

(ii) a family of homeomorphisms

$$\phi_i : U_i \to \Omega_i \subset \mathbb{R}^n \quad (\Omega_i \text{ is open})$$

such that $\forall\, i, j \in \Lambda$

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$

is a $C^k$ diffeomorphism.

Remark:

- $\phi_j \circ \phi_i^{-1}$, $i, j \in \Lambda$ (with $U_i \cap U_j \neq \emptyset$) are called transition functions.
- $(U_i, \phi_i)$ is a called a (coordinate) chart,
- $\phi_i^{-1} : \Omega_i \to U_i \subset M$ is a local parametrization.

Def: Two $C^k$ atlas for $M$, say $(U_i, \phi_i)_{i\in\Lambda_1}$ and $(V_j, \psi_j)_{j\in\Lambda_2}$, are $C^k$ equivalent if their union is still a $C^k$ atlas, that is, if $\forall i \in \Lambda_1$, $j \in \Lambda_2$ (st $U_i \cap V_j \neq \emptyset$)

$$\phi_i \circ \psi_j^{-1} : \psi_j(U_i \cap V_j) \to \phi_i(U_i \cap V_j)$$

are $C^k$ diffeomorphisms.

Def: A differentiable structure of class $C^k$ on $M$ is an equivalence class of $C^k$ atlas.

Remark: If $M$ is connected, then the integer $n$ in the definition does not depend on the chart and is defined as the dimension of $M$.

**Def**: A $C^k$ <u>differentiable manifold</u> of dimension $n$ is a pair $(M, \mathcal{A})$, where $M$ is a Hausdroff top. space and $\mathcal{A} = \{(U_i, \phi_i)\}_{i \in \Lambda}$ is a $C^k$ altas on $M$ with $\phi_i(U_i) \subset \mathbb{R}^n$.

<u>Remark</u>:
- In this course, we consider only $C^\infty$ differentiable manifold which is <u>connected</u> and a further condition such that "partitions of unity" is always possible.
- All compact manifolds satisfy the further condition.
- We'll refer such a manifold as a <u>smooth</u> manifold (or even simply <u>manifold</u>.)

eg: $M = T^n$, the n-torus $\left(T^n = \underbrace{S^1 \times \cdots \times S^1}_{n}\right)$

Let $f: \mathbb{R}^n \to T^n \subset \mathbb{C}^n$

$$\begin{array}{ccc} \mathbb{R}^n & \to & T^n \\ \cup & & \cup \\ (x_1, \cdots, x_n) & \mapsto & (e^{ix_1}, \cdots, e^{ix_n}) \end{array}$$

($f$ is onto)

$\forall\, p \in T^n$, $\exists\; x^p = (x_1^p, \cdots, x_n^p) \in \mathbb{R}^n$ s.t.

$p = f(x^p)$ (one may choose $x_i^p \in [0, 2\pi)$, $i = 1, \cdots, n$)

Consider $\Omega_p = (x_1^p - \pi, x_1^p + \pi) \times \cdots \times (x_n^p - \pi, x_n^p + \pi) \subset \mathbb{R}^n$

and let $\begin{cases} U_p = f(\Omega_p) \subset T^n & (U_p \text{ open \& contains } p) \\ \phi_p = (f|_{\Omega_p})^{-1} : U_p \to \Omega_p \subset \mathbb{R}^n \text{ homeo.} \end{cases}$

Then $\{(U_p, \phi_p)\}_{p \in T^n}$ is an $C^\infty$ altas on $T^n$:

In fact, if $p, q \in T^n$ s.t. $U_p \cap U_q \neq \emptyset$,

then $\phi_q \circ \phi_p^{-1}(x_1, \cdots, x_n)$ $\quad \left( (x_1, \cdots, x_n) \in \phi_p(U_p \cap U_q) \subset \Omega_p \right)$

$$
\begin{aligned}
&= \phi_q(f(x_1, \cdots, x_n)) \\
&= \phi_q(e^{ix_1}, \cdots, e^{ix_n}) \qquad \left( (e^{ix_1}, \cdots, e^{ix_n}) \in U_p \cap U_q \right) \\
&= (f|_{\Omega_q})^{-1}(e^{ix_1}, \cdots, e^{ix_n}) \\
&= (x_1 + 2k_1\pi, \cdots, x_n + 2k_n\pi) \quad \text{for some } k_1, \cdots, k_n
\end{aligned}
$$

s.t. $x_i + 2k_i\pi \in (x_i^q - \pi, x_i^q + \pi)$

note that $k_i$ are indep. of $(x_1, \cdots, x_n) \in \phi_p(U_p \cap U_q)$

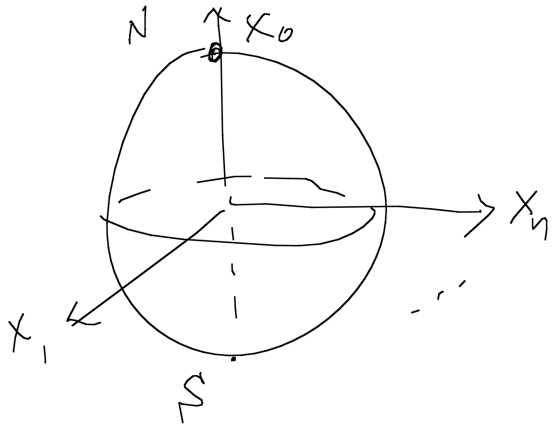
hence $\phi_q \circ \phi_p^{-1}$ is just a <u>translation</u> in $\mathbb{R}^n$.

Therefore $\phi_q \circ \phi_p^{-1}$ is a $C^\infty$ diffeo.

$\Rightarrow$ $(T^n, \{(U_p, \phi_p)\}_{p \in T^n})$ is a smooth manifold.

eg $M = S^n$, the n-sphere $S^n = \{(x_0, x_1, \dots, x_n) : \sum_{j=0}^n x_j^2 = 1\} \subset \mathbb{R}^{n+1}$

Let $\begin{cases} N = (1, 0, \dots, 0) \in S^n \\ S = (-1, 0, \dots, 0) \in S^n \end{cases}$




$\begin{cases} U_1 = S^n \setminus \{N\} \\ U_2 = S^n \setminus \{S\} \end{cases}$

$U_1 \cup U_2 = S^n$

Let

$$\left\{\begin{array}{l}\phi_1 : U_1 \to \mathbb{R}^n \qquad \text{(Stereographic projections)} \\ \quad \rotatebox{90}{$\in$} \\ (x_0, x_1, \cdots, x_n) \longmapsto \dfrac{1}{1-x_0}(x_1, \cdots, x_n) \\ \\ \phi_2 : U_2 \to \mathbb{R}^n \\ \quad \rotatebox{90}{$\in$} \\ (x_0, x_1, \cdots, x_n) \longmapsto \dfrac{1}{1+x_0}(x_1, \cdots, x_n)\end{array}\right.$$

are homeomorphisms.

Note that if $\phi_1(x_0, x_1, \cdots, x_n) = (y_1, \cdots, y_n)$

then

$$\phi_1^{-1}(y_1, \cdots, y_n) = \left(\frac{|y|^2-1}{|y|^2+1}, \frac{2y_1}{|y|^2+1}, \cdots, \frac{2y_n}{|y|^2+1}\right)$$

If $y \neq 0$

$$\phi_2 \circ \phi_1^{-1}(y_1, \cdots, y_n) = \frac{1}{1+\frac{|y|^2-1}{|y|^2+1}}\left(\frac{2y_1}{|y|^2+1}, \cdots, \frac{2y_n}{|y|^2+1}\right)$$

$$= \frac{1}{|y|^2}(y_1, \cdots, y_n)$$

In short $$\boxed{\phi_2 \circ \phi_1^{-1}(y) = \frac{y}{|y|^2} \quad \forall y \in \mathbb{R}^n \setminus \{0\}}$$

which is a $C^\infty$ diffeomorphism

$\Rightarrow \mathcal{A} = \{(U_1, \phi_1), (U_2, \phi_2)\}$ is an altas on $S^n$,

therefore $(S^n, \mathcal{A})$ is a smooth manifold.

<u>eg</u> $\mathbb{RP}^n$ the <u>real projective space</u> (in some book : $P^n\mathbb{R}$)

- As topological space

$\mathbb{RP}^n$ = quotient space of $\mathbb{R}^{n+1} \setminus \{0\}$ by the equivalent relation:

$$x \sim y \iff \exists \lambda \neq 0 \in \mathbb{R} \text{ s.t. } x = \lambda y$$

$(x, y \in \mathbb{R}^{n+1} \setminus \{0\})$

$= S^n / \{\pm Id\}$

(Hence $\mathbb{R}P^n$ is Hausdroff, compact, connected.)

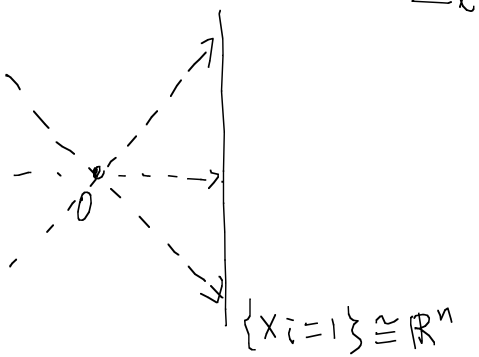
- Let $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}P^n$ be the canonical projection map

i.e. $\pi(x) =$ equi. class of $x$.

Define $V_i = \{ x = (x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1} : x_i \neq 0 \}$

$$\begin{aligned} \Phi_i : V_i &\longrightarrow \mathbb{R}^n \\ x &\longmapsto \left( \frac{x_0}{x_i}, \cdots, \widehat{\frac{x_i}{x_i}}, \cdots, \frac{x_n}{x_i} \right) \end{aligned}$$

this means the term deleted.




Then $\forall\, x, y \in V_i$, we have

$$(*)\quad \Phi_i(x) = \Phi_i(y) \iff \pi(x) = \pi(y) \quad (\text{i.e. } x \sim y)$$

(check!)

This gives

$$\begin{tikzcd} \mathbb{R}^{n+1}\setminus\{0\} \supseteq V_i \arrow[r, "\Phi_i"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{R}^n \\ U_i = \pi(V_i) \arrow[ur, "\phi_i"'] & \end{tikzcd}$$

where $\phi_i$ is defined by

$$\phi_i : U_i = \pi(V_i) \longrightarrow \mathbb{R}^n$$
$$\text{eq. class of } x \longmapsto \Phi_i(x)$$

($\phi_i$ is well-defined because of $(*)$.)

Using $\pi$: $\quad \phi_i(\pi(x)) = \Phi_i(x) \quad \sim \quad \phi_i \circ \pi = \Phi_i$

Further $\phi_i : U_i \to \mathbb{R}^n$ is homeomorphism (Check) with inverse

$$\phi_i^{-1}(y_0, \cdots, y_{n-1}) = \pi(y_0, \cdots, y_{i-1}, 1, y_i, \cdots, y_{n-1})$$

Therefore, if $y_j \neq 0$, for $j < i$, we have

$$(\phi_j \circ \phi_i^{-1})(y_0, \cdots, y_{n-1}) = \phi_j(\pi(y_0, \cdots, y_{i-1}, 1, y_i, \cdots, y_{n-1}))$$

$$= \Phi_j(y_0, \cdots y_j, \cdots, 1, y_i, \cdots, y_{n-1})$$

$$= \left(\frac{y_0}{y_j}, \cdots, \frac{\widehat{y_j}}{y_j}, \cdots, \frac{1}{y_j}, \frac{y_i}{y_j}, \cdots, \frac{y_{n-1}}{y_j}\right)$$

$\therefore$ $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is a $C^\infty$ diffeo,

Hence $\mathbb{RP}^n$ with the altas $\{(U_i, \phi_i)\}_{i=0}^{n}$ is a smooth manifold.

Note: $\mathbb{RP}^n$ is non-orientable for $n$ even according to the following definition: (proof omitted)

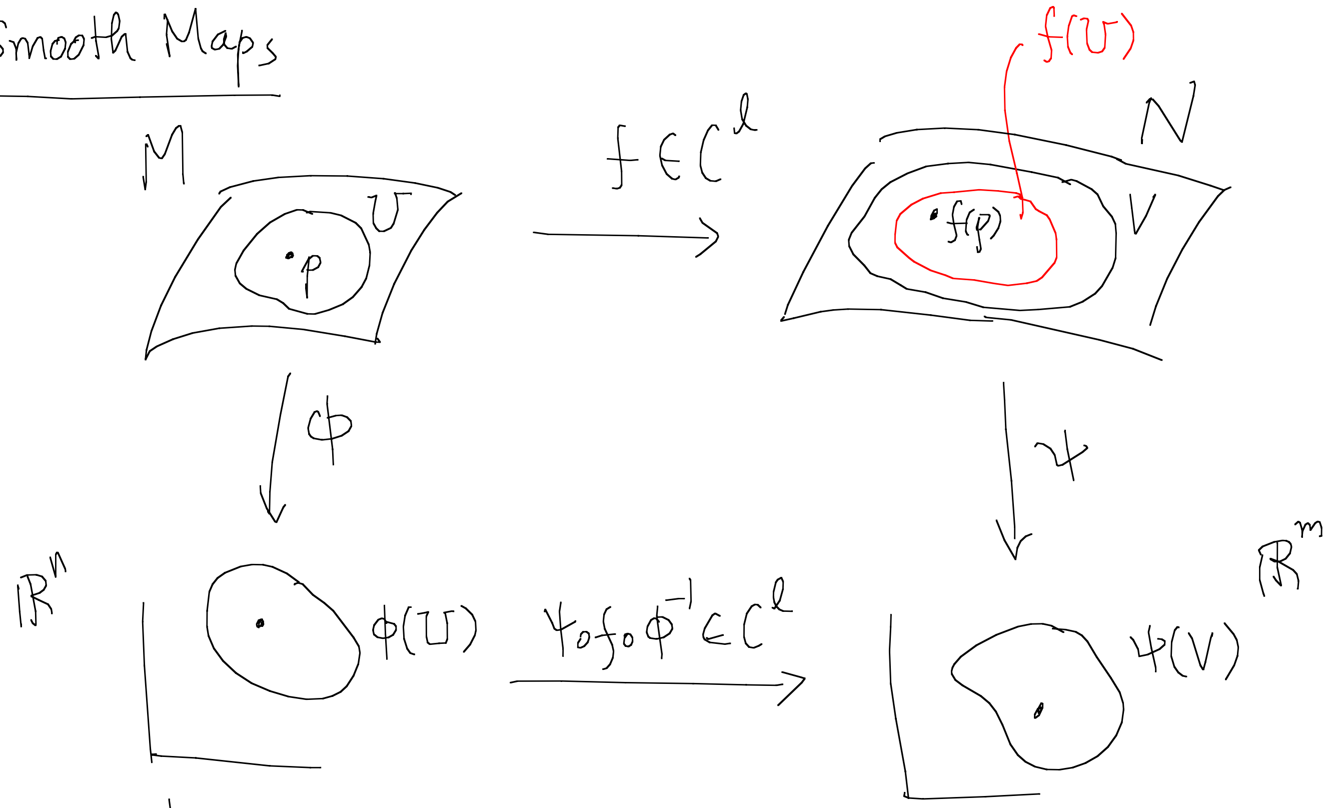
Def: A smooth manifold $M$ is said to be orientable if $\exists$ an atlas on $M$ s.t.

$$J(\phi_j \circ \phi_i^{-1}) > 0 \quad \forall\, i, j$$

$\uparrow$

Jacobian determinant of $\phi_j \circ \phi_i^{-1}$.

## 1.2 Smooth Maps




( $\psi \circ f \circ \phi^{-1}$ is just $m$ functions of $n$ variables as in Calculus )

Def: Let $M$ & $N$ be $C^k$ manifolds. A continuous map $f: M \to N$ is $\underline{C^\ell}$ map (for $\ell \le k$) if $\forall p \in M$, $\exists$ charts $(U, \phi)$ & $(V, \psi)$ for $M$ & $N$ around $p$ & $f(p)$ respectively with $f(U) \subset V$ such that

$$\psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V) \text{ is } C^\ell$$

Note: This definition does not depend on the charts since transition functions are $C^k$ $(k \ge \ell)$ (Ex!)

Def: A $C^k$ map $\gamma : (a,b) \to M$ from an interval to a smooth manifold is called a $\underline{C^k \text{ curve}}$ (on $M$).

Def: A $C^k$ map $f: M \to \mathbb{R}$ (or $\mathbb{C}$) is called a <u>$C^k$ function</u> on $M$.

Def: A smooth map $g: \mathbb{R}^n \to \mathbb{R}^k$ is a <u>submersion</u> (an <u>immersion</u>, a <u>local diffeomorphism</u>) at $x \in \mathbb{R}^n$ if $D_x g: \mathbb{R}^n \to \mathbb{R}^k$ is <u>surjective</u> (<u>injective</u>, bijective)

Def: Let $M$ & $N$ be smooth manifolds. A smooth map $f: M \to N$ is a <u>submersion</u> (<u>immersion</u>, <u>local diffeomorphism</u>) at $p \in M$, if $\exists$ charts $(U, \phi)$ for $M$ around $p$, $(V, \psi)$ for $N$ around $f(p)$

with $f(U) \subset V$ s.t. $\psi \circ f \circ \phi^{-1}$ is a <u>submersion</u>

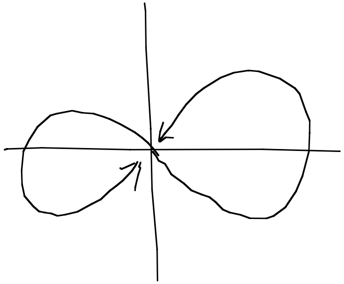
( <u>immersion</u>, <u>local diffeomorphism</u> ) at $\phi(p) \in \phi(U) \subset \mathbb{R}^n$.

<u>Def</u>: A map $f: M \to N$ is a <u>submersion</u> (<u>immersion</u>, <u>local diffeomorphism</u>) if it has the property at any point of $M$.

<u>Def</u>: A map $f: M \to N$ is a <u>diffeomorphism</u> if it is a <u>bijection</u> such that both <u>$f$ and $f^{-1}$ are smooth</u>.

<u>Def</u>: A map $f: M \to N$ is an <u>embedding</u> if it is an <u>immersion</u> and $f: M \to f(M) \subset N$ (with subspace top) is a <u>homeomorphism</u>.

__eg:__ $\mathbb{R} \xrightarrow{\gamma}$



$|\gamma'| \neq 0$

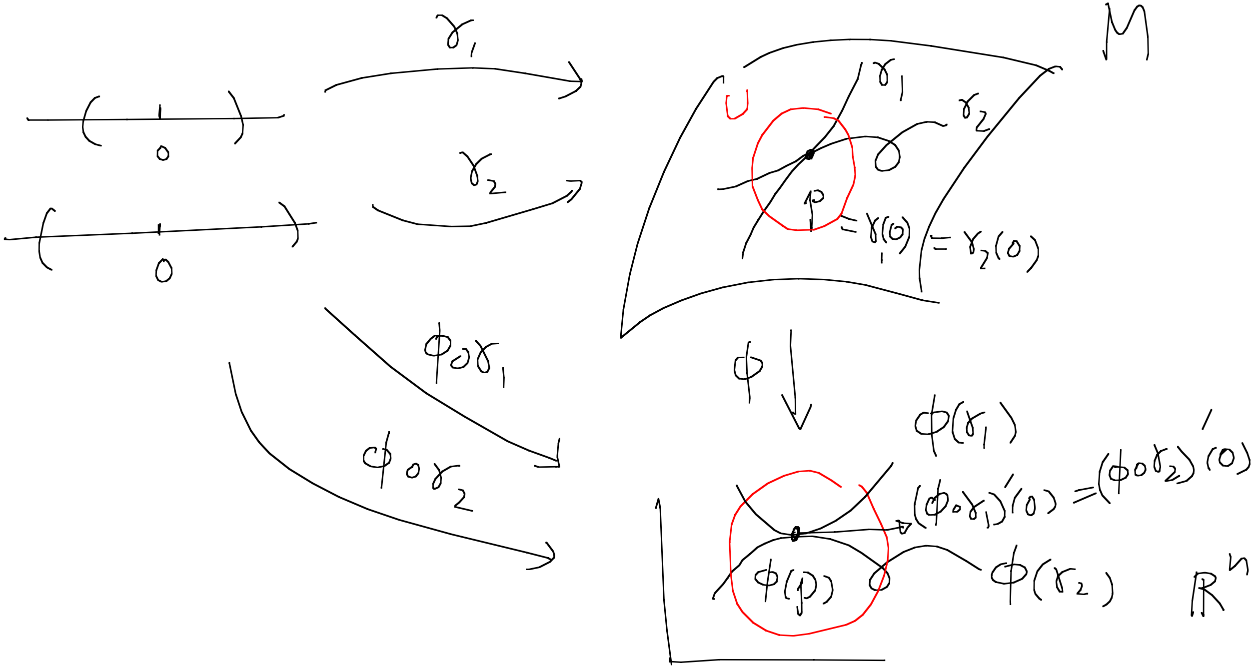
this is an immersion but not __embedding__ (Ex)

## 1.3 __Tangent vectors__

__Def 1:__ Let $M$ be a smooth manifold and $p \in M$.
A __tangent vector__ to $M$ at $p$ is an equi. class of $C^\infty$ curves $\gamma : I \to M$, where $I$ = interval containing $0$, such that $\gamma(0) = p$, for the equi. relation defined by $\gamma_1 \sim \gamma_2 \iff$

$$(\phi \circ \gamma_1)'(0) = (\phi \circ \gamma_2)'(0)$$

for a chart $(U, \phi)$ around $p$.




Ex: Check that the equi. relation is well-defined by showing that for any other chart $(V, \psi)$ around $p$, we have

$$(\psi\circ\gamma)'(0) = D_{\phi(p)}(\psi\circ\phi^{-1})\cdot(\phi\circ\gamma)'(0)$$

where $D_{\phi(p)}(\psi\circ\phi^{-1})$ is the Jacobi matrix (a differential) of the map $\psi\circ\phi^{-1}$ at $\phi(p)$.

Def 2 (Equivalent definition for tangent vectors)

Let $M$ be a smooth manifold, $p\in M$. $(U,\phi)$ & $(V,\psi)$ be 2 coordinates charts for $M$ around $p$. Let $u, v$ be 2 vectors in $\mathbb{R}^n$ (considered as tangent vectors to $\mathbb{R}^n$ at $\phi(p)$ & $\psi(p)$ respectively) We say that

$$\boxed{(U,\phi,u)\simeq(V,\psi,v) \Leftrightarrow D_{\phi(p)}(\psi\circ\phi^{-1})u = v}$$

Then a tangent vector to $M$ at $p$ is a equi. class of triples $(U, \phi, u)$.

<u>Note</u>: • In def 1, a tangent vector is represented by a curve $\gamma$. We usually write $\gamma'(0)$ for the tangent vector $[\gamma]$ for simplicity (Independent of charts)

• In def 2, the "same" tangent vector will be represented in a chart $(U, \phi)$ by a vector $u \in \mathbb{R}^n$.

• Def 1 $\Leftrightarrow$ Def 2 by taking $u = (\phi \circ \gamma)'(0)$.

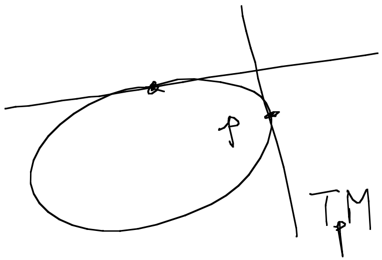
<u>Notation</u>: The set of tangent vectors to $M$ at $p$ is denoted by $T_pM$. (Tangent space to $M$ at $p \in M$).

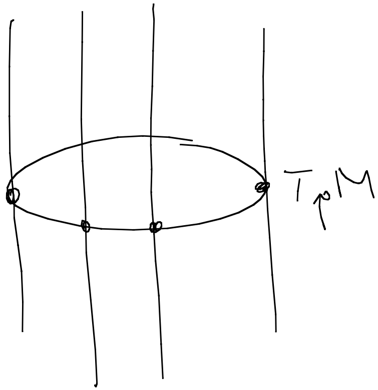
**Note:** If a chart $(U, \phi)$ is given, then we have an "isomorphism"

$$\begin{array}{rcl} \theta_{U,\phi,p} : \mathbb{R}^n & \longrightarrow & T_pM \\ u & \longmapsto & [(U,\phi,u)] \end{array}$$

(check: 1-1, onto)

**Def:** The *disjoint* union $TM$ of $T_pM$, $\forall p \in M$, is called the *tangent bundle* of $M$.



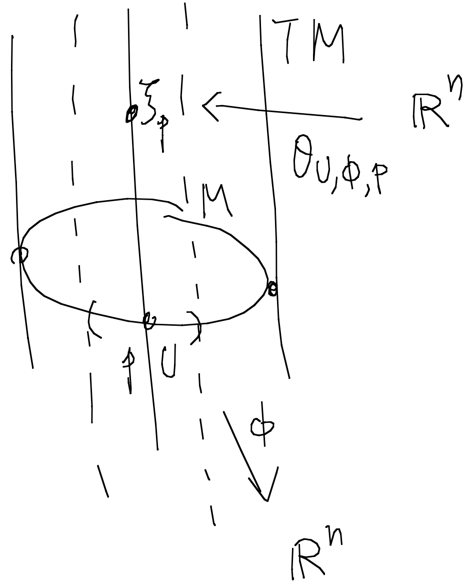



Thm: Let $M$ be an $n$-dim. $C^p$ manifold $(p>1)$. Then $TM$ can be equipped with a $\underline{2n}$-dim'l $\underline{C^{p-1}}$ abstract manifold structure.

Pf: (Sketch)

For each chart $(U, \phi)$ of $M$, define a "chart"

$$\left( \coprod_{p \in U} T_pM, \Phi \right) \text{ for } TM$$



by

$$\Phi(\xi_p) = \left(\phi(p),\ \theta^{-1}_{U,\phi,p}(\xi_p)\right) \in \mathbb{R}^n \times \mathbb{R}^n \cong \mathbb{R}^{2n}$$

$\forall\, \xi_p \in T_pM$, $p \in U$

Then one can see all these "$\coprod_{p\in U} T_pM$" give a topology on $TM$ such that $\Phi$ are homeomorphisms.

And one can check that $TM$ is Hausdroff and

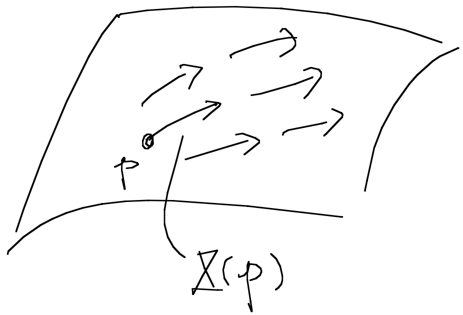
$\left\{ \left( \coprod_{p\in U} T_pM, \Phi \right) \right\}_{(U,\phi)}$ forms an $C^{p-1}$ altas of $TM$.

(we've differentiated once in the equiv. relation for tangent vectors)

**Def**: A (smooth) vector field $X$ on a manifold $M$ is a smooth section of the tangent bundle $TM$ of $M$,

i.e. $X : M \to TM$ is a smooth map

s.t. $X(p) \in T_pM$




- The set of vector fields on $M$ is denoted by $\Gamma(TM)$.