Department of Mathematics
The Chinese University of Hong Kong

MAT5061 Riemannian Geometry I
Final Examination

Apr 20, 2015

Answer all questions and show all your steps in detail.

(1) (20 marks)

(a) Define Levi-Civita connection (Riemannian connection) on a Riemannian manifold with metric $g = \langle \cdot, \cdot \rangle$.

(b) Proof the existence and uniqueness of Levi-Civita connection.

(2) (20 marks) Consider the Riemannian manifold defined by $M = (R^2_+, \frac{dx^2+dy^2}{y^2})$, where $R^2_+ = \{(x,y) \in \mathbb{R}^2 \,:\, y > 0\}$.

(a) Find all the Christoffel symbols of the Levi-Civita connection.

(b) Let $v_0 = (0,1)$ be considered as a tangent vector in $T_{(0,1)}M$ and $v(t)$ be the parallel transport of v_0 along the curve $\gamma(t) = (t,1)$, $-\infty < t < +\infty$. Show that $v(t)$ makes an angle t with the y-direction, measured in the **Euclidean and clockwise** sense.

(3) (20 marks)

(a) Let $\mathbb{S}^2 \times \mathbb{S}^2$ be the submanifold of $\mathbb{R}^6$ defined by

$$\{(x_1, x_2, x_3, y_1, y_2, y_3) \,:\, x_1^2 + x_2^2 + x_3^2 = y_1^2 + y_2^2 + y_3^2 = 1\}.$$

Prove that the sectional curvature of the Riemannian manifold $\mathbb{S}^2 \times \mathbb{S}^2$ with induced metric is non-negative.

(b) Find a totally geodesic flat torus embedded in $\mathbb{S}^2 \times \mathbb{S}^2$.

(4) (20 marks) Let $\gamma : [0, b] \to M$ be a normalized geodesic with $\gamma(0) = x$ and $\gamma'(0) = v$. Suppose that J is the Jacobi field along γ such that $J(0) = 0$ and $J'(0) = w$ with $|w| = 1$ and $\langle w, v \rangle = 0$. Find, in terms of the sectional curvature $K(\pi)$ of the 2-plane section π generated by v and w at x, the Taylor expansion of $|J(t)|^2$ about $t = 0$ up to order 4 .

(5) (20 marks) Suppose that M is a Riemannian manifold.

(a) Show that for any $x \in M$, the differential $(d \exp_x)_0$ of $\exp_x$ at the origin can be identified as the identity map of the tangent space $T_x M$.

(b) Using (a), show that for any $x \in M$, there exists a neighborhood U of x and a number $\delta > 0$ such that, for any $y \in U$, the restriction $\exp_y |_{B(\delta)}$ of $\exp_y$ on the open δ-ball centered at the origin $B(\delta) \subset T_y M$ is a diffeomorphism.

(End)