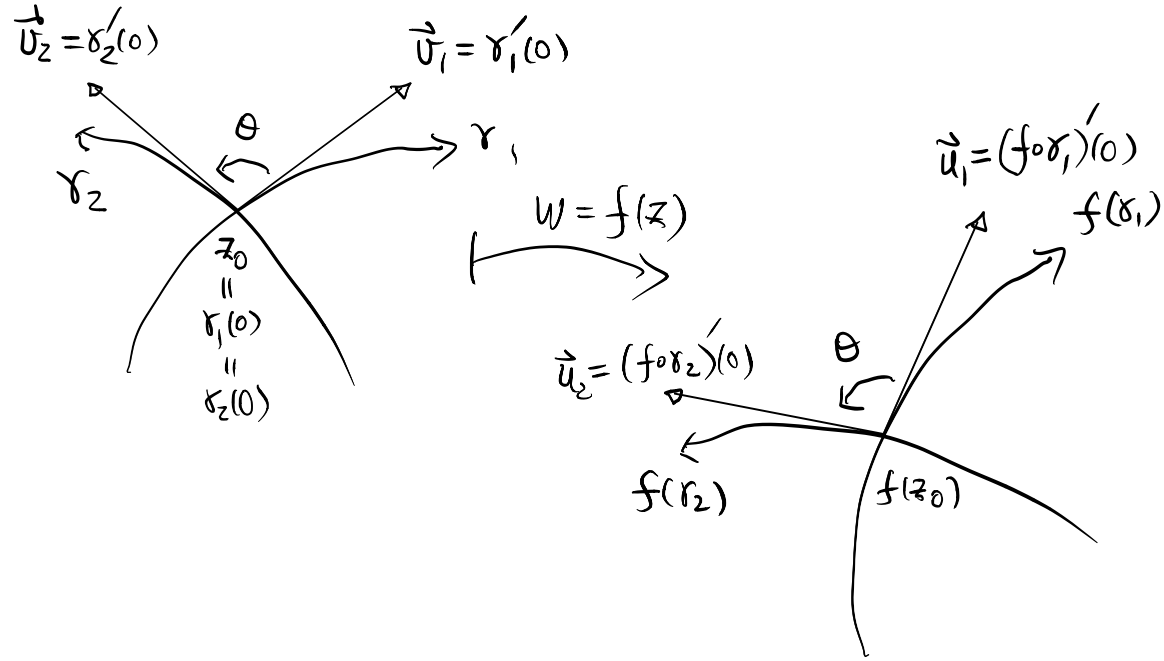
Ch6 Conformal Mapping & Riemannian Mapping Theorem

§6.1 Conformal Mapping

Def: (1) A transformation $W = f(z)$ is said to be conformal at a point $z_0 \in \mathbb{C}$ if it is locally one-to-one and preserves angles at z_0 in following sense:

$\forall$ two curves $\gamma_1(t)$ and $\gamma_2(t)$ passing through z_0 at $t=0$, i.e. $\gamma_1(0) = \gamma_2(0) = z_0$,

the angle between $f \circ \gamma_1$ and $f \circ \gamma_2$ at $f(z_0)$ equals the angle between γ_1 and γ_2 at z_0



(2) If $w=f(z)$ is defined on a domain D and conformal at z_0 for any $z_0 \in D$, then $w=f(z)$ is called a conformal mapping on D.

Thm 1: Let f be a function analytic at z_0. Then $f'(z_0) \neq 0$ implies f is conformal at z_0.

Pf: By Cor 2 in §5.5, f is locally 1-1 at z_0. So we only need to show that f preserves angle at z_0.

Notation as in the above figure.

Note that

$$\vec{u}_i = (f \circ \gamma_i)'(0) = f'(\gamma_i(0))\,\gamma_i'(0)$$

$$= f'(z_0)\,\vec{v}_i \quad (\text{for } i=1,2)$$

$f'(z_0) \neq 0 \Rightarrow f'(z_0) = |f'(z_0)|\,e^{i\theta_0}$ for some $\theta_0 \in \arg f'(z_0)$

$\therefore \quad \vec{u}_i = |f'(z_0)|\,e^{i\theta_0}\,\vec{v}_i$

Hence $\vec{u}_i$ are obtained by scaling and rotating the $\vec{v}_i$ (respectively) wrt the same factor and angle.

$\therefore$ the angle from $\vec{u}_1$ to $\vec{u}_2$ equals the angle from $\vec{v}_1$ to $\vec{v}_2$. ※

<u>Remarks</u> (1) $f'(z_0) \neq 0 \Rightarrow f$ is invertible near z_0.

$\therefore \exists\, g(w)$ defined near $w_0 = f(z_0)$ s.t.

$$g(f(z)) = z \quad \& \quad f(g(w)) = w.$$

And one can show that $g(w)$ is also analytic at w_0 and

$$\boxed{g'(w) = \frac{1}{f'(g(w))}} \quad \forall\, w \text{ near } w_0$$

(Pf: Ex!)

(2) Hence if a conformal mapping $f: D \to \Omega$ is actually (globally) one-to-one, then $g = f^{-1}: \Omega \to D$ exists and also conformal.

<u>egs</u> (1) <u>Rotation</u>: $w = e^{i\theta_0} z \quad (\theta_0 \in \mathbb{R})$

clearly 1-1, $\frac{dw}{dz} = e^{i\theta_0} \neq 0 \Rightarrow$ conformal

Inverse: $z = e^{-i\theta_0} w$ is also a rotation.

(2) Scaling: $w = rz \quad (r > 0)$

clearly 1-1, $\frac{dw}{dz} = r \neq 0 \Rightarrow$ conformal

Inverse: $z = \frac{1}{r} w$ is also a scaling.

(3) translation: $w = z + b \quad (b \in \mathbb{C})$

Clearly 1-1, $\frac{dw}{dz} = 1 \neq 0 \quad \therefore$ conformal

Inverse: $z = w + (-b)$ is also a translation.

(4) (Complex) inversion $\quad w = \frac{1}{z} \quad (z, w \in \mathbb{C} \setminus \{0\})$

clearly 1-1, $\frac{dw}{dz} = -\frac{1}{z^2} \neq 0 \quad \forall z \in \mathbb{C} \setminus \{0\}$.

$\therefore$ conformal

Inverse: $z = \frac{1}{w}$ is also the inverse.

More generally, we have

Linear Fractional Transformations (Möbius Transformations)

$$\boxed{w = \frac{az+b}{cz+d} \quad (ad - bc \neq 0)}$$

(where $a, b, c, d \in \mathbb{C}$)

(1) Domain of definition: $w = \frac{az+b}{cz+d}$ is not defined

at $z = -\frac{d}{c}$.

(2) <u>Extension to $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$</u>

Using $\lim_{z \to -\frac{d}{c}} \frac{az+b}{cz+d} = \infty$, and

$$\lim_{z \to \infty} \frac{az+b}{cz+d} = \frac{a}{c} \quad \left(\text{could be } \infty, \text{ when } c=0\right)$$

$\left(\frac{a}{c} \text{ is well-defined as } a, c \text{ cannot be both zero as } ad-bc \neq 0. \right)$,

we define for $z \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

$$w = \begin{cases} \frac{az+b}{cz+d}, & \text{if } z \neq -\frac{d}{c}, \infty \\ \infty, & \text{if } z = -\frac{d}{c} \\ \frac{a}{c}, & \text{if } z = \infty. \end{cases}$$

Then $z \mapsto w \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ can be considered as a mapping from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$.

And we'll use the <u>same notation</u>

$$w = \frac{az+b}{cz+d} \quad (ad-bc \neq 0) \quad a, b, c, d \in \mathbb{C}$$

to denote the extended mapping.

(3) $w = \frac{az+b}{cz+d} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is 1-1 and the inverse

is $z = \frac{(-d)w+b}{cw+(-a)}$ which is also a linear

fractional transformation $\left((-d)(-a)-bc = ad-bc \neq 0\right)$

(4) $w = \frac{az+b}{cz+d}$ $(ad-bc \neq 0)$ is conformal as

$$\frac{dw}{dz} = \frac{ad-bc}{(cz+d)^2} \neq 0 \quad \left(\text{for } z \in \mathbb{C} \setminus \{-\tfrac{d}{c}\}\right)$$

(One can justify the conformality at $z = -\frac{d}{c}$ & ∞ in a more advanced discussion.)

(5) If $c=0$, then $ad-bc \neq 0 \Rightarrow ad \neq 0 \Rightarrow a, d \neq 0$

and $w = \left(\frac{a}{d}\right) z + \frac{b}{d}$

$$= \left|\frac{a}{d}\right| e^{i \arg\left(\frac{a}{d}\right)} z + \left(\frac{b}{d}\right)$$

$\therefore$ w is obtained from z by a <u>rotation</u>, then <u>scaling</u>, then a <u>translation</u>.

If $c \neq 0$, then

$$w = \frac{az+b}{cz+d} = \frac{a}{c} + \frac{-(ad-bc)}{c} \frac{1}{(cz+d)} \quad (\text{Ex!})$$

$\therefore$ w is obtained by a rotation, then a scaling, then a translation, then a inversion, then another rotation, scaling and translation.

In all cases, a linear fractional transformation is a composition of rotations, scalings, translations, and inversion.

(6) Composition of 2 linear fractional transformations is also a linear fractional transformation.

(Pf: Ex!)

(7) Linear fractional transformation maps lines to lines or circles, and maps circles to lines, or circles.

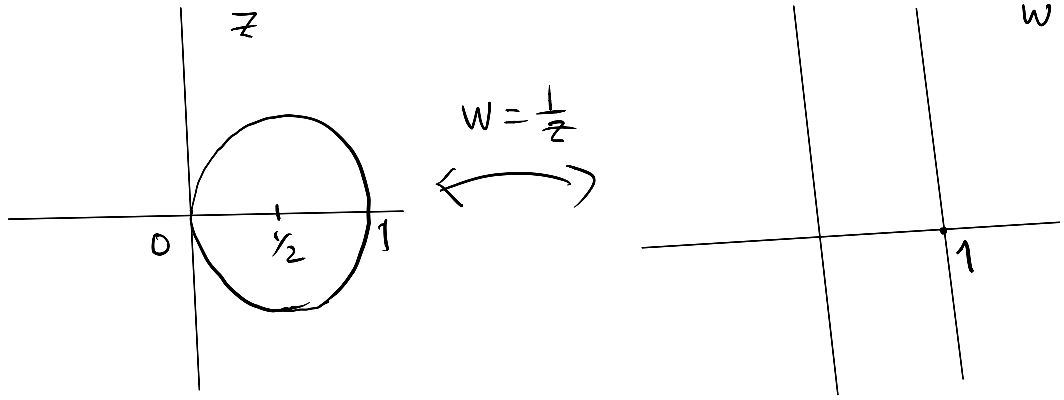
$$\{\text{lines, circles}\} \xleftrightarrow[\text{transformations.}]{\text{linear fractional}} \{\text{lines, circles}\}$$

<u>Pf</u>: Clearly true for rotations, scalings & translations.

It is also true for inversion (Ex!)

$\Rightarrow$ ture for compositions of rotations, scalings, translations & inversion $\#$

eg5



for $w = 1 + iv$ ($v \in \mathbb{R}$)

then $z = \left(\frac{1}{1+v^2}\right) + \left(-\frac{v}{1+v^2}\right)i$

$\Rightarrow \quad |z - \frac{1}{2}|^2 = \frac{1}{4}$

$\therefore$ line $\{1 + iv : v \in \mathbb{R}\} \longleftrightarrow$ circle $\{|z - \frac{1}{2}|^2 = \frac{1}{4}\}$.

(8) Given any two sets of distinct three points $\{z_1, z_2, z_3\}$ and $\{w_1, w_2, w_3\}$ of $\hat{\mathbb{C}}$, there exists a unique linear fractional transformation f such that

$$f(z_i) = w_i \quad \text{for } i = 1, 2, 3.$$

Pf: Special case $w_1 = 0,\ w_2 = 1,\ w_3 = \infty$

$$T(z) = \frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1}$$

Need to check $ad - bc \neq 0$: let $a = \frac{z_2 - z_3}{z_2 - z_1} \neq 0, \infty$

$$T(z) = \frac{az + (-az_1)}{(1)z + (-z_3)} \quad \& \quad \begin{aligned} & a(-z_3) - 1(-az_1) \\ & = a(z_1 - z_3) \neq 0 . \end{aligned}$$

is the required linear fractional transformation.

<u>General case</u>: By special case, $\exists$ linear fractional transformations T and W s.t.

$$T : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \quad \text{takes } \{z_1, z_2, z_3\} \text{ to } \{0, 1, \infty\}$$

$$W : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \quad \text{takes } \{w_1, w_2, w_3\} \text{ to } \{0, 1, \infty\}$$

Then $f = W^{-1} \circ T$ is a linear fractional transformation taking $\{z_1, z_2, z_3\}$ to $\{w_1, w_2, w_3\}$.

<u>Uniqueness</u>: Let f, g be 2 linear fractional transformations taking $\{z_1, z_2, z_3\}$ to $\{w_1, w_2, w_3\}$. Then

$$S = T \circ g^{-1} \circ f \circ T^{-1}$$ is a linear fractional transformation

where T = linear fractional transformation in the special case.

It is easy to check
$$\begin{cases} S(0) = 0 \\ S(1) = 1 \\ S(\infty) = \infty \end{cases} \qquad (\text{Ex!})$$

Suppose $S(z) = \frac{az+b}{cz+d}$ $(ad-bc \neq 0)$

Then $0 = \frac{b}{d}$, $1 = \frac{a+b}{c+d}$, $\infty = \frac{a}{c}$

$\therefore$ $b=0$, $c=0$ (as $a,b,c,d \in \mathbb{C}$)

Substitute into $1 = \frac{a+b}{c+d} \Rightarrow 1 = \frac{a}{d}$

$\therefore$ $S(z) = (\frac{a}{d}) z = z$ is the identity transformation.

$$\Rightarrow T \circ g^{-1} \circ f \circ T^{-1} = Id$$

$$\Rightarrow g^{-1} \circ f = T^{-1} \circ T = Id$$

$$\Rightarrow f = g \quad \#$$

(9) Cross-ratio

The expression $\frac{z-z_1}{z-z_3} \cdot \frac{z_2-z_3}{z_2-z_1}$ in the proof of the special case of (8) is called the cross-ratio of the four points z, z_1, z_2, z_3 and usually denoted by

$$\boxed{(z, z_1, z_2, z_3) = \frac{z-z_1}{z-z_3} \cdot \frac{z_2-z_3}{z_2-z_1}}$$

(Caution: Some define the cross-ratio in different order.

For instance, some textbooks denote

$$cr(z, z_2; z_1, z_3) = \frac{z-z_1}{z-z_3} \cdot \frac{z_2-z_3}{z_2-z_1} \quad)$$

(10) Using cross-ratio, the linear fractional transformation taking $\{z_1, z_2, z_3\}$ to $\{w_1, w_2, w_3\}$ is given implicitly by

$$\boxed{(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)}$$

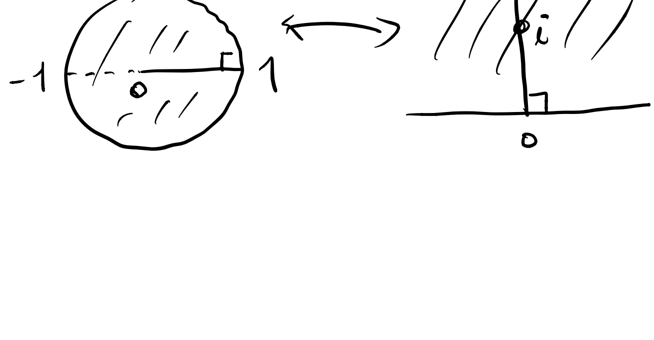
(11) Proof of (10) can be given by observing that $\forall$ linear fractional transformation f,

$$\boxed{\big(f(z), f(z_1), f(z_2), f(z_3)\big) = (z, z_1, z_2, z_3)}$$

i.e. cross-ratio is <u>invariant</u> under linear fractional transformations.

(Pf: Ex!)

<u>eg6</u>: A conformal map from $\{|z|<1\}$ 1-1 onto $\{x+iy : y>0\}$



We have many choice. Let us find the linear fractional transformation that takes $\{1, 0, -1\}$ to $\{0, i, \infty\}$.

By (10), it is given implicitly by

$$(w, 0, i, \infty) = (z, 1, 0, -1)$$

i.e.
$$\frac{w-0}{w-\infty} \cdot \frac{i-\infty}{i-0} = \frac{z-1}{z-(-1)} \cdot \frac{0-(-1)}{0-1}$$

(in limiting sense as $w_3 \to \infty$)

$$\therefore \quad \frac{w}{i} = \frac{1-z}{1+z}$$

i.e.
$$w = i\,\frac{1-z}{1+z}.$$

Note that for $|z|=1$, i.e. $z = e^{i\theta}$, we have

$$w = i\,\frac{1-e^{i\theta}}{1+e^{i\theta}} = \frac{\sin\theta}{1+\cos\theta} \in \mathbb{R}$$

$\therefore$ $w = i\,\frac{1-z}{1+z}$ maps the unit circle $|z|=1$ to the x-axis, and maps the interior point 0 to the upper half-plane. Therefore, by the fact that $w = i\frac{1-z}{1+z}$ is a diffeomorphism (1-1, onto, differentiable, and with differentiable inverse),

$$w = i\frac{1-z}{1+z} \text{ maps } \{|z|<1\} \text{ to } \{x+iy : y>0\}.$$

§6.2 An Application of Conformal Maps

In many application, one need to solve the following boundary value problem of the Laplace equation:
(u real-valued)

$$\begin{cases} \Delta u \overset{def}{=} u_{xx}+u_{yy} = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = \varphi & (\text{a given function}) \end{cases}$$

Def 1 : Solutions of Laplace equation are called <u>harmonic functions</u>.

Thm 1 : Real part and imaginary part of an analytic function are harmonic.

(Pf : Ex!)

Thm 2 : If h is harmonic and f is analytic function. Then $g = h\circ f$ is harmonic.

Pf : Let $f = u+iv$ and $h = h(u,v)$.

h harmonic $\Leftrightarrow$ $h_{uu}+h_{vv} = 0$

Now $\quad g_x = h_u u_x + h_v v_x$

$$g_{xx} = (h_{uu} u_x + h_{uv} v_x) u_x + h_u u_{xx} + (h_{vu} u_x + h_{vv} v_x) v_x + h_v v_{xx}$$

Similarly $\quad g_{yy} = (h_{uu} u_y + h_{uv} v_y) u_y + h_u u_{yy} + (h_{vu} u_y + h_{vv} v_y) v_y + h_v v_{yy}$

$$\therefore \Delta g = h_{uu}(u_x^2 + u_y^2) + 2 h_{uv} (\cancelto{0}{u_x v_x + u_y v_y}) + h_{vv}(v_x^2 + v_y^2) + h_u (\cancelto{0}{u_{xx} + u_{yy}}) + h_v (\cancelto{0}{v_{xx} + v_{yy}})$$

CR eqt $\Rightarrow$

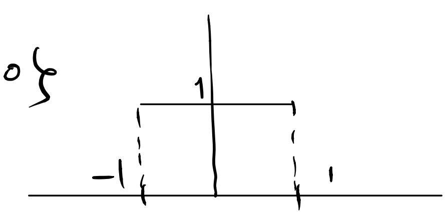
$$\begin{cases} u_x^2 + u_y^2 = v_x^2 + v_y^2 \\ u_x v_x + u_y v_y = 0 \\ \Delta u = \Delta v = 0 \quad (\text{Thm } 1) \end{cases}$$

$$\therefore \Delta g = (\cancelto{0}{h_{uu} + h_{vv}})(u_x^2 + u_y^2) = 0 \quad \#$$

By Thm 2, there is a hope that one can use a conformal mapping f to transform the (possibly difficult to handle) domain Ω to a domain $f(\Omega)$ so that the corresponding problem is easier to solve.

__eg 1__

$$\begin{cases} u_{xx} + u_{yy} = 0 \quad \text{on } \{x + iy : y > 0\} \\ u(x, 0) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases} \end{cases}$$



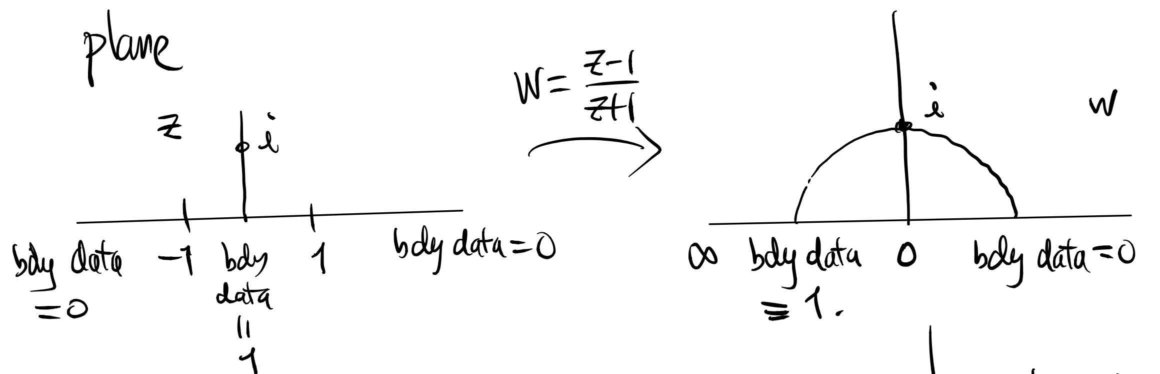
First, we move the discontinuity $1, -1$ of $u(x,0)$ to $0, \infty$ by $w = \dfrac{z-1}{z+1}$.

Note that $1 \mapsto 0$, $-1 \mapsto \infty$ (and $0 \mapsto -1$)

We see that $w = \dfrac{z-1}{z+1}$ maps the x-axis to x-axis.

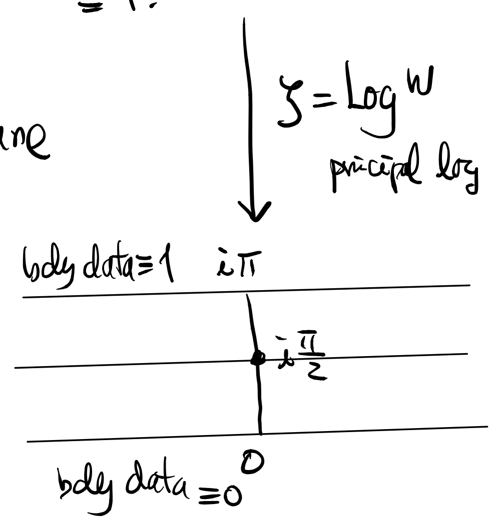
Note also that $i \mapsto i$.

$\therefore$ $w = \dfrac{z-1}{z+1}$ maps upper half-plane to upper half-plane



Then we maps the upper half-plane to the strip

$\{ \zeta = \xi + i\eta : 0 < \eta < \pi \}$



Consider the bdy value problem

$\Delta \varphi = 0$

$$\left\{\begin{array}{l}\varphi(\xi,\pi)=1 \\ \varphi(\xi,0)=0\end{array}\right. \qquad \forall \xi\in\mathbb{R}$$

This is easily solved by (the imaginary part of $\frac{\xi}{\pi}$)

$$\varphi(\xi,\eta)=\frac{1}{\pi}\eta$$

Hence the original problem has the solution

$$u(x,y)=\varphi(\xi(x,y),\eta(x,y))=\frac{1}{\pi}\eta(x,y)$$

where $(x,y)\mapsto \operatorname{Log}\left(\dfrac{x+iy-1}{x+iy+1}\right)=\xi+i\eta$.

$$\therefore \quad u(x,y)=\frac{1}{\pi}\operatorname{Arg}\left(\frac{x+iy-1}{x+iy+1}\right) \qquad \text{(principal argument)}$$

$$=\frac{1}{\pi}\operatorname{Arg}\left(\frac{(x^2+y^2-1)+2iy}{(1+x)^2+y^2}\right)$$

$$=\frac{1}{\pi}\tan^{-1}\left(\frac{2y}{x^2+y^2-1}\right) \qquad (0\leq \tan^{-1}t\leq\pi)$$

is the required solution.