HW5 Due Mar 23, 2017

1. Let $R > 0$ be the radius of convergence of $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ at z_0. Show that by term-by-term differentiation and mathematical induction,

$$f^{(n)}(z) = \sum_{k=0}^{\infty} \frac{(n+k)!}{k!} a_{n+k} (z-z_0)^k \qquad (n=0,1,2,\cdots)$$

for $|z-z_0| < R$.

2. Use multiplication of series to show that

$$\frac{e^z}{z(z^2+1)} = \frac{1}{z} + 1 - \frac{1}{2}z - \frac{5}{6}z^2 + \cdots \qquad (0 < |z| < 1)$$

3. Show that $f(z) = 1 - \cos z$ has a zero of order 2 at $z_0 = 0$.

4. Let f be an analytic function in a domain D with zeros $z_1, z_2, \cdots, z_n$ ($z_i \neq z_j$ if $i \neq j$) of respective order $m_1, m_2, \cdots, m_n$. Show that there exists an analytic function $g(z)$ on D such that

$$f(z) = (z-z_1)^{m_1} (z-z_2)^{m_2} \cdots (z-z_n)^{m_n} g(z).$$

5. Let f be an entire function such that for $x \in \mathbb{R}$, $f(x) = \sum_{k=0}^{\infty} a_k x^k$ (a convergence power series, a_k may be complex). Show that $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $\forall z \in \mathbb{C}$.

6. Suppose that D is a domain symmetric with respect to the reflection across the real axis $\mathbb{R}$ and $D \cap \mathbb{R} = (a,b)$ is an interval. Suppose also that f is an analytic function on D. Show that

(a) $g(z) = \overline{f(\bar{z})}$ is analytic on D, and

(b) $g(z) = f(z)$, $\forall z \in D$ if and only if

$f(x) \in \mathbb{R}$, $\forall x \in (a,b)$.