§3.8 Cauchy Integral Formula

Thm 1 Let f be analytic everywhere inside and on a simple closed contour C in positive orientation. If z_0 is any point interior to C, then

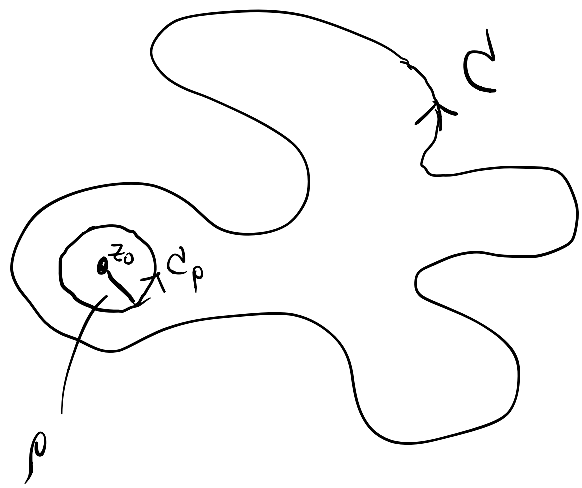
$$\boxed{f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz}$$

Cauchy Integral Formula.

Pf: Since z_0 is interior to C, for any sufficiently small ρ,

$D_\rho(z_0) = \{|z-z_0| \le \rho\}$

is interior to C.



Let $C_\rho = \partial D_\rho(z_0) = \{|z-z_0| = \rho\}$ in positive orientation parametrized by

$$z = z_0 + \rho e^{i\theta}, \quad 0 \le \theta \le 2\pi$$

Then principle of deformation of paths $\Rightarrow$

$$\int_C \frac{f(z)}{z-z_0}\,dz = \int_{C_\rho} \frac{f(z)}{z-z_0}\,dz, \quad \forall \text{ sufficiently small } \rho > 0.$$

$$= \int_0^{2\pi} \frac{f(z_0+\rho e^{i\theta})}{\rho e^{i\theta}} \cdot i\rho e^{i\theta}\, d\theta$$

$$= i\int_0^{2\pi} f(z_0+\rho e^{i\theta})\, d\theta .$$

As f analytic at $z_0 \Rightarrow f$ continuous at z_0

$\Rightarrow \forall \varepsilon > 0, \exists \rho_0 > 0$ s.t.

$$|f(z_0+\rho e^{i\theta}) - f(z_0)| < \varepsilon, \ \forall\, 0<\rho<\rho_0.$$

$$\therefore \left| \int_C \frac{f(z)}{z-z_0}\, dz - 2\pi i f(z_0) \right|$$

$$= \left| i\int_0^{2\pi} f(z_0+\rho e^{i\theta})\, d\theta - i\int_0^{2\pi} f(z_0)\, d\theta \right|$$

$$\leq \int_0^{2\pi} |f(z_0+\rho e^{i\theta}) - f(z_0)|\, d\theta$$

$< 2\pi\varepsilon$, for ρ sufficiently small & $0<\rho<\rho_0$.

$$\therefore \int_C \frac{f(z)}{z-z_0}\, dz = 2\pi i f(z_0).$$

$\times\!\!\times$

<u>Notation</u>: (i) $f(z) = \dfrac{1}{2\pi i} \displaystyle\int_C \frac{f(s)}{s-z}\, ds$

in order to let z moves inside C.

(ii) $f^{(n)}(z)$ denotes the n-th derivative of f at z (if exists), where $f^{(0)}(z) = f(z)$.

$$f^{(n)}(z) = \frac{d}{dz} f^{(n-1)}(z).$$

Thm2 Let f be analytic inside and on a simple closed contour C taken in the positive orientation. If z is any point interior to C, then $\forall\, n = 0, 1, 2, \cdots$,

$$\boxed{f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(s)}{(s-z)^{n+1}}\, ds}$$

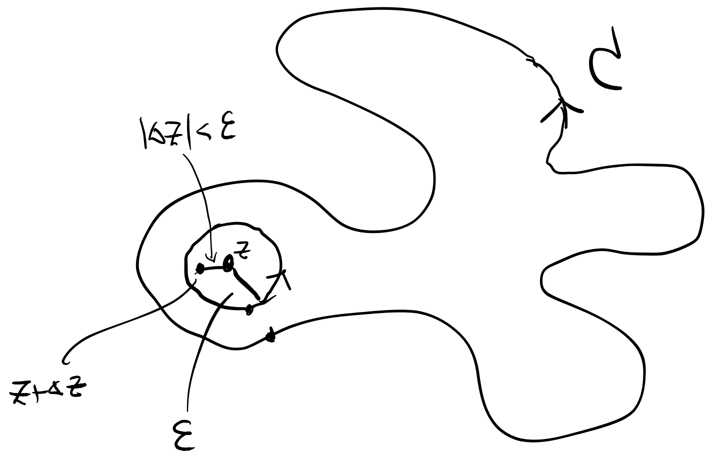
(Cauchy Integral Formula)

Proof: The case of $n=0$ is proved in Thm 1. Assume the thm is true for n. Then

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(s)}{(s-z)^{n+1}}\, ds$$

and $$f^{(n)}(z+\Delta z) = \frac{n!}{2\pi i} \int_C \frac{f(s)}{(s-z-\Delta z)^{n+1}}\, ds$$

for $|\Delta z| < \varepsilon$ with $\varepsilon > 0$ small enough so that $D_\varepsilon(z) = \{ |s - z| \le \varepsilon \}$ interior to C



$$\therefore\ f^{(n)}(z+\Delta z) - f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \left[\frac{1}{(s-z-\Delta z)^{n+1}} - \frac{1}{(s-z)^{n+1}}\right] f(s)\,ds$$

$$= \frac{n!}{2\pi i}\int_C \frac{(s-z)^{n+1} - (s-z)^{n+1} + (n+1)(s-z)^n \Delta z + \sum_{k=2}^{n+1} (-1)^k \binom{n+1}{k} (s-z)^{n+1-k} (\Delta z)^k}{(s-z-\Delta z)^{n+1}(s-z)^{n+1}} f(s)\,ds$$

$$= \frac{n!}{2\pi i}\int_C \frac{(n+1)(s-z)^n \Delta z}{(s-z-\Delta z)^{n+1}(s-z)^{n+1}} f(s)\,ds$$

$$+ \frac{n!}{2\pi i}\int_C \frac{(\Delta z)^2 \left[\sum_{k=2}^{n+1} (-1)^k \binom{n+1}{k} (s-z)^{n+1-k} (\Delta z)^{k-2}\right]}{(s-z-\Delta z)^{n+1}(s-z)^{n+1}} f(s)\,ds$$

$$= \Delta z \cdot \frac{(n+1)!}{2\pi i}\int_C \frac{f(s)}{(s-z-\Delta z)^{n+1}(s-z)}\,ds + (\Delta z)^2 \cdot I$$

where $$I = \frac{n!}{2\pi i}\int_C \frac{\left[\sum_{k=2}^{n+1} (-1)^k \binom{n+1}{k} (s-z)^{n+1-k} (\Delta z)^{k-2}\right]}{(s-z-\Delta z)^{n+1}(s-z)^{n+1}} f(s)\,ds$$

$$\therefore \quad \frac{f^{(n)}(z+\Delta z)-f^{(n)}(z)}{\Delta z} - \frac{(n+1)!}{2\pi i}\int_C \frac{f(s)}{(s-z)^{n+2}}\,ds$$

$$= \frac{(n+1)!}{2\pi i}\int_C \left[\frac{1}{(s-z-\Delta z)^{n+1}(s-z)} - \frac{1}{(s-z)^{n+2}}\right] f(s)\,ds + \Delta z \cdot \mathrm{I}$$

$$= \Delta z \cdot \mathrm{I} + \frac{(n+1)!}{2\pi i}\int_C \frac{(s-z)^{n+1} - (s-z-\Delta z)^{n+1}}{(s-z-\Delta z)^{n+1}(s-z)^{n+2}} f(s)\,ds$$

$$= \Delta z \cdot \mathrm{I} + \frac{(n+1)!}{2\pi i}\int_C \frac{\Delta z \cdot \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} (s-z)^{n+1-k} (\Delta z)^{k-1}}{(s-z-\Delta z)^{n+1}(s-z)^{n+2}} f(s)\,ds$$

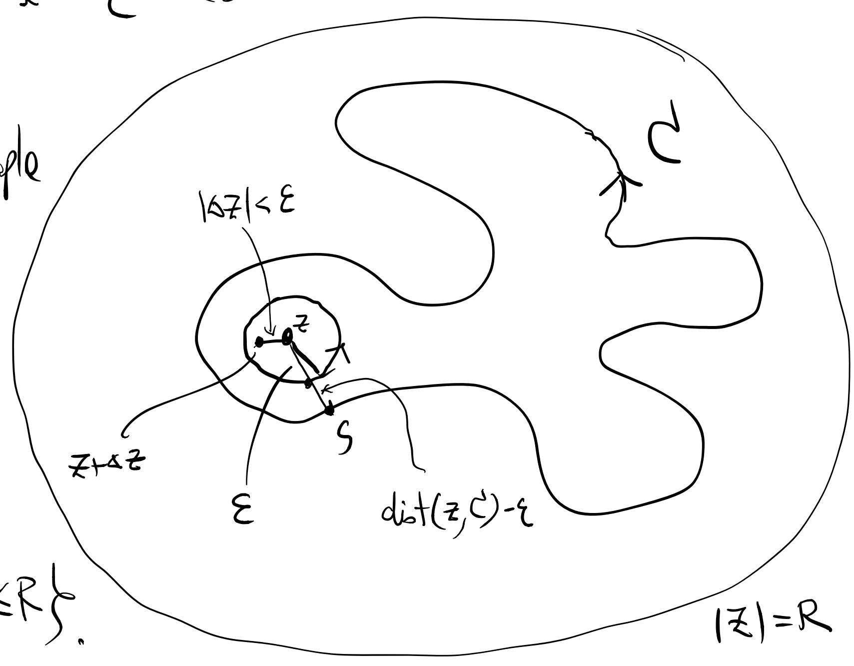
$$= \Delta z\,(\mathrm{I} + \mathrm{II})$$

where $$\mathrm{II} = \frac{(n+1)!}{2\pi i}\int_C \frac{\sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} (s-z)^{n+1-k} (\Delta z)^{k-1}}{(s-z-\Delta z)^{n+1}(s-z)^{n+2}} f(s)\,ds$$

Now since the simple closed contour C is compact,

$\exists\, R > 0$

s.t. C is enclosed in $\{|z| \le R\}$.



Hence $|s-z| \le |s| + |z| \le 2R$.

On the other hand, $\{|s-z| \leq \varepsilon\}$ is interior to C,

$\forall s \in C$, $\quad |s-z| > \varepsilon > 0$, and

$$|s-(z+\Delta z)| \geq \operatorname{dist}(z, C) - \varepsilon > 0 .$$

Let $M = \sup_{s \in C} |f(s)|$ and $L =$ length of C.

Then

$$|I| \leq \frac{n!}{2\pi} \int_C \frac{\left|\sum_{k=2}^{n+1} (-1)^k \binom{n+1}{k} (s-z)^{n+1-k} (\Delta z)^{k-2}\right|}{|s-z-\Delta z|^{n+1} |s-z|^{n+1}} |f(s)| |ds|$$

$$\leq \frac{n!}{2\pi} \cdot \frac{\sum_{k=2}^{n+1} \binom{n+1}{k} (2R)^{n+1-k} \varepsilon^{k-2}}{(\operatorname{dist}(z, C) - \varepsilon)^{n+1} \varepsilon^{n+1}} \cdot ML \quad (\text{for } |\Delta z| < \varepsilon)$$

Similarly

$$|II| \leq \frac{(n+1)!}{2\pi} \frac{\sum_{k=1}^{n+1} \binom{n+1}{k} (2R)^{n+1-k} \varepsilon^{k-1}}{(\operatorname{dist}(z, C) - \varepsilon)^{n+1} \varepsilon^{n+2}} \cdot ML \quad (\text{for } |\Delta z| < \varepsilon)$$

$\therefore$ there exists $C = C(z, R, \varepsilon, M, L, n) > 0$ independent of Δz (with $|\Delta z| < \varepsilon$) such that

$$\left| \frac{f^{(n)}(z+\Delta z) - f^{(n)}(z)}{\Delta z} - \frac{(n+1)!}{2\pi i} \int_C \frac{f(s)\, ds}{(s-z)^{n+2}} \right| \leq |\Delta z| \left(|II| + |III|\right)$$

$$\leq C\, |\Delta z| .$$

$$\Rightarrow \quad f^{(n+1)}(z) = \frac{(n+1)!}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^{n+2}} \quad \text{by letting } |\Delta z|\to 0.$$

Hence induction $\Rightarrow$ the thm is true for all $n=0,1,2,\cdots$. #

§3.9 Some Consequences of Cauchy Integral Formula, Liouville's Theorem and Fundamental Theorem of Algebra

Thm 1 If a function f is analytic at a given point, then its derivatives of all order are analytic there too.

Pf: f analytic at z_0

$\xRightarrow{\text{by def}}$ f analytic in a nbd. $\{|z-z_0|<\varepsilon\}$ of z_0.

$\Rightarrow$ f analytic inside and on the circle

$$C_{\frac{\varepsilon}{2}} : |z-z_0| = \frac{\varepsilon}{2}$$

Then Cauchy Integral Formula

$$\Rightarrow \quad f^{(2)}(z) = \frac{2!}{2\pi i}\int_{C_{\frac{\varepsilon}{2}}} \frac{f(s)}{(s-z)^3}\,ds$$

exist for all $z \in \{|z-z_0|<\frac{\varepsilon}{2}\}$

$\Rightarrow$ $f^{(1)}(z)$ is analytic in $\{|z-z_0|<\frac{\varepsilon}{2}\}$

i.e. $f'(z)$ is analytic at z_0.

Similar argument with math. induction

$\Longrightarrow \quad f^{(n)}$ is analytic at z_0, $\forall n$. ※

Cor: If a function $f(z)=u(x,y)+iv(x,y)$ is analytic at a point $z=x+iy$, then u and v have continuous partial derivatives of all order at that point.

Thm2: Let f be continuous on a domain D.

If $\int_C f(z)=0$ for every closed contour C in D,

then f is analytic throughout D.

Pf: If $\int_C f(z)dz=0$ $\forall$ closed contour C in D

then f has an antiderivative F in D,

i.e. $F'(z)=f(z)$, $\forall z\in D$.

$\Rightarrow$ F is analytic in D

$\therefore$ Thm1 $\Rightarrow$ $f=F'$ is analytic $\forall z\in D$ ※

Thm3 (Cauchy Inequality) Suppose that a function f is analytic inside and on a positively oriented circle C_R, centered at z_0 with radius R. If M_R denotes the maximum value of $|f(z)|$ on C_R, then

$$\boxed{|f^{(n)}(z_0)| \le \frac{n!\, M_R}{R^n}} \quad (\forall n=1,2,3,\cdots)$$

Pf: By Cauchy Integral Formula,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{C_R} \frac{f(s)\,ds}{(s-z_0)^{n+1}}, \quad n=1,2,3,\cdots$$

Since $|s-z_0|=R$, $\forall s \in C_R$,

and length of $C_R = 2\pi R$,

we have

$$|f^{(n)}(z_0)| \le \frac{n!}{2\pi} \cdot \frac{M_R}{R^{n+1}} \cdot 2\pi R = \frac{n!\, M_R}{R^n}.$$

\#

Thm 4 (Liouville's Theorem) If a function f is entire and bounded in the complex plane, then $f(z)$ is constant throughout the plane.

Pf: Let M be the bound of f, i.e. $|f(z)| \le M$, $\forall z \in \mathbb{C}$.

Then $\forall z_0 \in \mathbb{C}$ and any $R>0$,

f entire $\Rightarrow$ f analytic inside & on $C_R = \{|z-z_0|=R\}$

$\Rightarrow$ (Cauchy Inequality, $n=1$)

$$|f'(z_0)| \le \frac{M_R}{R} \le \frac{M}{R}.$$

Since $R>0$ is arbitrary, we have $f'(z_0)=0$ (by letting $R \to +\infty$). $\therefore f' \equiv 0$ on $\mathbb{C}$

$\Rightarrow f =$ constant function. #

Thm 5 (Fundamental Theorem of Algebra)

Any polynomial $P(z) = a_0 + a_1 z + \cdots + a_n z^n$, $(a_n \neq 0)$ of degree n $(n \ge 1)$ has at least one zero.

(i.e. $\exists\, z_0 \in \mathbb{C}$ s.t. $P(z_0)=0$).

Pf: Suppose not. Then $P(z) \neq 0$, $\forall z \in \mathbb{C}$

$\Rightarrow$ $f(z) = \dfrac{1}{P(z)}$ is an entire function.

Now for $z \neq 0$,
$$P(z) = a_0 + a_1 z + \cdots + a_n z^n = z^n\left(a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right)$$

For $k = 0, 1, \cdots, n-1$, $\left|\dfrac{a_k}{z^{n-k}}\right| \le \dfrac{|a_k|}{|z|^{n-k}} < \dfrac{|a_n|}{2n}$

if $|z| > \sqrt[n-k]{\dfrac{2n|a_k|}{|a_n|}}$.

Therefore, for $R = \max\limits_{k=0,1,\cdots,n-1} \left(\sqrt[n-k]{\dfrac{2n|a_k|}{|a_n|}} \right) > 0$,

we have for $|z|>R$, $\left|\dfrac{a_k}{z^{n-k}}\right| < \dfrac{|a_n|}{2n}$, $\forall k=0,1,\cdots,n-1$

$$\Rightarrow \left| \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right| \leq \left|\frac{a_{n-1}}{z}\right| + \cdots + \left|\frac{a_0}{z^n}\right| < n \cdot \frac{|a_n|}{2n} = \frac{|a_n|}{2}$$

$$\Rightarrow |P(z)| = |z^n| \left| a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right|$$

$$\geq |z|^n \left| a_n - \left| \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right| \right|$$

$$\geq |z|^n \frac{|a_n|}{2} > \frac{|a_n|}{2} R^n \quad \text{for } |z|>R$$

$$\therefore \quad |f(z)| = \left| \frac{1}{P(z)} \right| \leq \frac{2}{|a_n| R^n}, \quad \forall |z|>R.$$

Note that $P(z) \neq 0$, $\forall z \in \mathbb{C} \Rightarrow |f(z)| = \left|\dfrac{1}{P(z)}\right|$

is continuous on $\{|z| \leq R\}$ (closed & bounded)

$\Rightarrow \exists M_1 > 0$ s.t. $|f(z)| \leq M_1$, $\forall z \in \{|z| \leq R\}$.

Hence $|f(z)| \leq \max\{M_1, \frac{2}{|a_n| R^n}\}$, $\forall z \in \mathbb{C}$.

$\therefore$ f is a bounded entire function.

Liouville's Thm $\Rightarrow$ $f(z) = \frac{1}{P(z)} = $ constant.

Which contradiction the assumption that $n \geq 1$.

$\therefore$ $P(z)$ has a zero. ※

Note: By fundamental thm of algebra, we immediately have $P(z) = a_n (z - z_1)(z - z_2) \cdots (z - z_n)$ where $z_1, \cdots, z_n$ are zeroes (may not distinct) of $P(z)$.

§3.10 Maximum Modulus Principle

Lemma 1 (Gauss' mean value theorem)

If $f(z)$ is analytic inside and on $C_\rho = \{|z - z_0| = \rho\}$

then
$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{i\theta}) d\theta.$$

Pf: Cauchy Integral Famula

$$f(z_0) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z)}{z - z_0} dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + \rho e^{i\theta})}{\rho e^{i\theta}} i\rho e^{i\theta} d\theta$$
$$= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{i\theta}) d\theta \quad ※$$

<u>Lemma 2</u> Suppose that f is analytic in $\{|z-z_0|<\varepsilon\}$ and
$$|f(z)| \le |f(z_0)|, \quad \forall z \in \{|z-z_0|<\varepsilon\}$$
Then $f(z) \equiv f(z_0)$, $\forall z \in \{|z-z_0|<\varepsilon\}$.

<u>Pf</u>: $\forall\, 0<\rho<\varepsilon$, Gauss' mean value thm
$$\Rightarrow \quad f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+\rho e^{i\theta})\,d\theta$$
$$\Rightarrow \quad |f(z_0)| \le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\rho e^{i\theta})|\,d\theta$$
$$\le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|\,d\theta = |f(z_0)|$$
$$\therefore \quad |f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\rho e^{i\theta})|\,d\theta$$
i.e. $$\int_0^{2\pi}\Big[|f(z_0)| - |f(z_0+\rho e^{i\theta})|\Big]\,d\theta = 0.$$
Since $|f(z_0)| - |f(z_0+\rho e^{i\theta})| \ge 0$, $\forall \theta \in [0,2\pi]$,

we have $$|f(z_0)| \equiv |f(z_0+\rho e^{i\theta})|, \quad \forall \theta \in [0,2\pi].$$
Since $\rho \in (0,\varepsilon)$ is arbitrary, we have
$$|f(z_0)| \equiv |f(z)|, \quad \forall z \in \{|z-z_0|<\varepsilon\}$$
Recall that an analytic function with constant modulus is a constant. $\therefore$ $f(z) \equiv f(z_0)$. ※

Thm 1 (Maximum Modulus Principle)

If a function f is analytic and not constant in a given domain D, then $|f(z)|$ has no maximum value in D, i.e. $\nexists$ no point $z_0 \in D$ such that $|f(z)| \leq |f(z_0)|$ $\forall z \in D$.

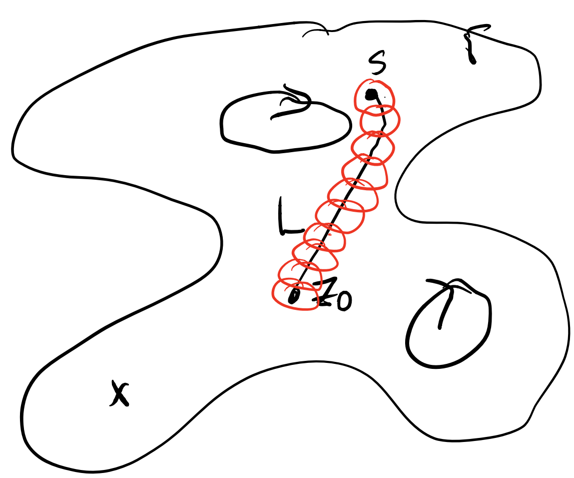
Note: This is equivalent to: If f analytic in a domain D and $\exists z_0 \in D$ such that

$$|f(z)| \leq |f(z_0)|, \quad \forall z \in D.$$

Then f is a constant function.

Pf: Suppose $\exists z_0 \in D$ s.t. $|f(z)| \leq |f(z_0)|$, $\forall z \in D$.

Then $\forall s \in D$, connect z_0 to s by a contour L in D (one may choose L be a polygonal line).



Let d = distance from L to ∂D.

Then (by compactness of L) there exist finitely many points

$z_1, \cdots z_{n-1}$ such that $D_d(z_k)$ interior to D, $\forall k$,

$D_d(z_0) \cup D_d(z_1) \cup \cdots \cup D_d(z_{n-1}) \supset L$

and $z_1 \in D_d(z_0)$, $z_2 \in D_d(z_1)$, $\cdots$, $s \in D_d(z_{n-1})$

where $D_d(z_k) = \{ |z - z_k| < d \}$.

Then applying lemma 2 to $D_d(z_0) \Rightarrow f(z_1) = f(z_0)$

$\Rightarrow |f(z_1)| \geq |f(z)|$, $\forall z \in D_d(z_1)$.

$\Rightarrow f(z_2) = f(z_1) = f(z_0)$ & $|f(z_2)| \geq |f(z)|$ $\forall z \in D_d(z_2)$

$\cdots$

$\Rightarrow f(s) = f(z_{n-1}) = \cdots = f(z_0)$. ※

Cor Suppose that a function f is continuous on a closed and bounded region R and that it is analytic and not constant in the interior of R. Then the maximum value of $|f(z)|$ in R, which is always reach, occurs somewhere on the boundary of R and never in the interior.

(Pf: Immediately from maximum modulus principle.)